Library

FEB 15 1943

Mathematical Reviews

Vol. 4, No. 2 February, 1943 pp. 33–64

TABLE OF CONTENTS

Number theory 33

Analysis 36

Fourier series and generalizations, integral transforms 36

Polynomials, polynomial approximations . 40

Differential equations 42

Calculus of variations 46

Geometry 49

Algebraic geometry 51

Differential geometry 53

Relativity 55

Mechanics 57

Astronomy 57

Hydrodynamics, aerodynamics, acoustics . 58

Theory of elasticity 61

THE REPRINT SERVICE

The subscribers to MATHEMATICAL REVIEWS can obtain, either on microfilm or as photoprint, the complete text of any article of which an abstract is printed, except books, or material of which the reproduction is prohibited by copyright law. The microfilm requires some sort of reading machine or projector in which standard 35 mm. double perforated film can be viewed. Only *positive* prints are furnished; that is, the projected image appears as black words on a white background. Photoprints, on the other hand, are on sheets 10" by 14", and are similar to photostats. They can be read directly without the aid of any reading device. They also are furnished as positives, that is, black words on a white background.

Articles not abstracted in the MATHEMATICAL REVIEWS will be copied at the rates mentioned below provided the material is available in the library of Brown University. All microfilm copies of articles other than those with MF numbers will be supplied as negatives.

Cost

The cost of microfilm is 2¢ per exposure, *each exposure* (except in the case of a very few unusually large journals) *containing two pages of text.*

Photoprints cost 16¢ per sheet, *each sheet* (except in the case of unusually large journals) *containing two pages of text.*

The minimum charge for any one order (but not for any one item when several items are ordered simultaneously) is 50¢.

How to Order

The process of ordering either microfilm or photoprint may be made clear by considering an example. Each abstract bears a serial number easily recognized by the fact that it is always preceded by the abbreviation "MF". A subscriber who wishes the text of item 18 would first note that the original article occupied pages 590 to 608. It therefore covers 19 pages, and will require ten exposures of microfilm, or ten sheets of photoprint for its reproduction. Hence the microfilm charge for this particular item would be 20¢, and the photoprint charge $1.60. The charges so computed apply on any order totaling 50¢ or more; if the total is less than 50¢, the charge for the entire order will be 50¢.

In ordering:

1. Give your name and address.
2. State the serial number of the article or articles you wish.
3. Say whether you wish microfilm or photoprint.
4. Enclose cash for the amount of the order, either in currency, stamps, or by check.
5. Mail to MATHEMATICAL REVIEWS, Brown University, Providence, Rhode Island.

In the case of large journals, only one page of which can be accommodated on a microfilm exposure or a photoprint sheet, you will be notified of the extra charge before the order is filled, unless you have indicated on your order that the reprint is desired even at the higher price.

Why Cash Must Accompany Order

Individual orders for either microfilm or photoprint are so small that if credit books were kept and bills rendered, the extra clerical expenses would be a large proportion of the total. By requiring payment in advance, these charges have been eliminated, and the price to the customer correspondingly reduced. However, libraries and institutions which make extensive use of the service may be granted the privilege of payment at stated intervals.

ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

In order to supply persons interested in making card files for subject indices of their own fields, or to add bibliographical remarks to the REVIEWS in the future, there will be available an edition of MATHEMATICAL REVIEWS printed on only one side of the paper. This special edition may be obtained for an additional payment of $1.00 (that is, $14.00 for subscriptions, and $7.50 for members of sponsoring organizations). This edition will be folded but not stitched. Subscriptions for this edition should be entered as soon as possible since only a limited number can be printed. For the first four issues, only the two-side edition is available. A current regular subscription can be changed to the one-side subscription by informing MATHEMATICAL REVIEWS and paying $1.00.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
ACADEMIA NACIONAL DE CIENCIAS EXACTAS, FISICAS Y NATURALES DE LIMA
HET WISKUNDIG GENOOTSCHAP IN AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY
UNION MATEMATICA ARGENTINA

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

Mathematical Reviews

Vol. 4, No. 2 FEBRUARY, 1943 Pages 33-64

NUMBER THEORY

Bose, R. C. and Nair, K. R. On complete sets of Latin squares. Sankhyā 5, 361–382 (1941). [MF 7355]

Let the $m+1$ points on one particular line of a finite projective geometry be named X, Y, U_1, U_2, $\cdots$, U_{m-1}, and let the remaining m lines through each of these points be numbered $0, 1, \cdots, m-1$ in any order. Then line number s of the pencil through X meets line number t of the pencil through Y in a definite point P_{st}, which may be represented by the cell (s, t) of an $m \times m$ square. (This means column number s, row number t.) The first of a set of $m-1$ mutually orthogonal Latin squares may be constructed by entering in this cell the number j which is carried by that line of the pencil through U_1 which passes through the point P_{st}. The other Latin squares are derived similarly from the pencils through $U_2, \cdots, U_{m-1}$. If the geometry is $PG(2, m)$, where m is a prime power p^n, the ith Latin square (in standard form) is determined by the simple formula

$$l_j = l_s + l_i l_t, \tag{1}$$

where $l_0=0$, $l_1=1$, and $l_2, \cdots, l_{m-1}$ are the remaining elements of the Galois field $GF(m)$ in any particular order. Thus the number s appears in cell $(s, 0)$, and the number i in cell $(0, 1)$. [W. W. R. Ball, Mathematical Recreations and Essays, Macmillan Company, New York, 1939, p. 191, second footnote.] By considering automorphisms of the Galois field, it is deduced that the number of complete sets of mutually orthogonal Latin squares associated with $PG(2, p^n)$ is $(p^n-2)!/n$. From known enumerations of Latin squares of orders 2, 3, 4, 5, and 7, it follows that this method covers all such sets of squares with $m \leqq 7$, and that there exists no non-Desarguesian geometry with fewer than nine points on each line.

It is found that the various Latin squares of a set associated with $PG(2, p^n)$ are derivable from one another by permuting whole rows. The set is said to be in canonical form if it is derivable from one of its squares by cyclic permutation of the last $m-1$ rows (while row number 0 remains constantly $0, 1, \cdots, m-1$). Any such set can be obtained from (1) by putting $l_r = y^{r-1}$, where y is a primitive element of the Galois field. A certain set of 9×9 Latin squares in the Statistical Tables for Biological, Agricultural and Medical Research of Fisher and Yates [Oliver and Boyd, London, 1938] is shown to be associated with a certain non-Desarguesian geometry (with ten points on each line). In fact, the entries of the ith square are given by the formula $l_j = l_s + l_i l_t^{\epsilon}$, where $\epsilon=1$ or 3 according as the element l_i of $GF(3^2)$ is a square or a nonsquare. [Cf. also F. W. Levi, Finite Geometrical Systems, University of Calcutta, 1942; these Rev. 4, 49.] *H. S. M. Coxeter.*

Bose, R. C. On some new series of balanced incomplete block designs. Bull. Calcutta Math. Soc. 34, 17–31 (1942). [MF 7164]

In a previous paper [Ann. Eugenics 9, 353–399 (1939); these Rev. 1, 199], the author used the method of symmetrically repeated differences (which is a development of perfect partitions) to construct designs with $k=3$, $\lambda=1$ or 2, and with $k=4$ or 5, $\lambda=1$. He now extends that method to cover the following cases: (i) $k=4$ or 5, $\lambda=2$, $r=kt+2$, provided $2t+1$ is a prime power; (ii) $k=4$ or 5, $\lambda=2$, $r=kt$, provided $(\frac{k}{2})t+1$ $(=v)$ is a prime power; (iii) $k=4$ or 5, $\lambda=k-1$, $r=kt$, provided $kt+1$ $(=v)$ is a prime power. Most of these designs are new; for example, (i) with $k=5$ and $t=1$ is one of those listed by Fisher and Yates [Statistical Tables, Oliver and Boyd, London, 1938, p. 49] as "not yet solved." *H. S. M. Coxeter* (Toronto, Ont.).

Bose, R. C. An affine analogue of Singer's theorem. J. Indian Math. Soc. (N.S.) 6, 1–15 (1942). [MF 7231]

The author observes that the p^{2n} elements of the field $GF(p^{2n})$ are uniquely expressible in the form $x\lambda+y$, where λ is a given primitive element while x and y belong to the subfield $GF(p^n)$ generated by λ^{p^n+1}. He represents the element $\alpha = x\lambda+y$ by the point $(\alpha)=(x, y)$ of a finite affine geometry, and shows that this geometry is $EG(2, p^n)$. The points collinear with (α_1) and (α_2) are given by $(\theta\alpha_1+\varphi\alpha_2)$, where $\theta+\varphi=1$ and φ belongs to $GF(p^n)$. Letting (λ^{d_i}) $(i=1, 2, \cdots, p^n)$ denote the points collinear with $(1)=(0, 1)$ and $(\lambda)=(1, 0)$, he deduces the following theorem: If m is a power of a prime, we can find m integers d_i $(i=1, 2, \cdots, m)$ such that among the $m(m-1)$ differences d_i-d_j $(i \neq j)$, reduced modulo m^2-1, all the positive integers less than m^2-1 and not divisible by $m+1$ occur exactly once. Conversely, any such set of integers provides a concise representation for $EG(2, m)$. This resembles a theorem of James Singer [Trans. Amer. Math. Soc. 43, 377–385 (1938)], who uses $GF(p^{3n})$ instead of $GF(p^{2n})$. *H. S. M. Coxeter.*

Gupta, Hansraj. On numbers of the form $4^a(8b+7)$. J. Indian Math. Soc. (N.S.) 5, 192–202 (1941). [MF 6982]

This paper is concerned with the number $k(n)$ of integers not exceeding n which are not sums of less than 4 squares. By considering the digits of n when written to the base 4 it is readily shown that for $s \geqq 0$ and for each m such that $1 \leqq m \leqq 4s$, there exists an n such that $4^{s-1} \leqq n < 4^s$ for which $m = n - 6k(n)$. From this result it follows that for the function $F(n) = (n/6 - k(n))/\log n$, $\liminf F(n) = 0$, $\limsup F(n) = 1/\log 8$, and that every point between these limits is a limit point of $F(n)$. These three facts were proved previously by M. C. Chakrabarti [Bull. Calcutta Math. Soc. 32, 1–6 (1940); these Rev. 3, 162]. The rest of the paper considers the number $N(m, s)$ of values of $n < 4^s$ for which $n - 6k(n) = m$. In particular cases, such as $m = 4s-1$, in which $N(4s-1, s)$ is the sth term of the Fibonacci series, explicit results are obtained. *D. H. Lehmer.*

Patterson, J. O. A note on the Diophantine problem of finding four biquadrates whose sum is a biquadrate. Bull. Amer. Math. Soc. 48, 736–737 (1942). [MF 7274]

In 1753 Euler conjectured the existence of 4 biquadrates whose sum is a biquadrate; 158 years later Norrie discovered that

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4.$$

This note gives the second example:

$$240^4+340^4+430^4+599^4=651^4$$

and describes an exclusion method by which the author plans to search for further solutions of this Diophantine equation. *D. H. Lehmer* (Berkeley, Calif.).

Bell, E. T. Parametric solutions of certain Diophantine equations. Duke Math. J. 9, 431–435 (1942). [MF 6880]

One of the Diophantine equations considered by the author is $a_1x_1\cdots x_{i_1}+a_2y_1\cdots y_{i_2}+\cdots+a_nz_1\cdots z_{i_n}$ in the $i_1+i_2+\cdots+i_n$ variables. The solution of this is reduced to that of $\sum_{i=1}^{n}x_iy_i=0$ (whose complete parametric solution is known) and an associated multiplicative system. The other equation is $Q(x_1,\cdots,x_n)=\sum_{i=1}^{n}a_ix_iy_i$, where Q is the general homogeneous quadratic form in n variables with integral coefficients. To solve this last equation the author first finds the complete integral solution with all the $a_i=1$. It is also pointed out how a parametric solution of $Q(x_1,\cdots,x_{n-1})=uv$ may be obtained. [The special case $n=3$ has already been considered by Dickson and Latimer.] *I. A. Barnett* (Cincinnati, Ohio).

Niven, Ivan. Quadratic Diophantine equations in the rational and quadratic fields. Trans. Amer. Math. Soc. 52, 1–11 (1942). [MF 6990]

The author studies the number of integral solutions of the general quadratic equation

$$ax^2+bxy+cy^2+dx+ey+f=0,$$

where coefficients and solutions are from the field of rational numbers or from some quadratic field. Thus if $A=b^2-4ac<0$ and the field is that of the rational numbers, there is at most a finite number of solutions; if $A\geqq 0$ and there is one solution, then there must be an infinite number, aside from certain exceptional cases that are stated in detail. Similarly detailed results are given for real quadratic fields (in this case a finite number of solutions occurs if A is totally negative) and for imaginary quadratic fields (here one solution implies infinitely many, aside from certain exceptions). The exceptions essentially deal with the cases where the left hand side of the equation can be factored. As might be expected the whole discussion depends upon the study of the generalization of the Pell equation in quadratic fields. In this connection it is proved that the equation $\xi^2-\gamma\eta^2=1$ ($\gamma\neq 0$ an integer in a quadratic field F) has an infinite number of integral solutions in F if and only if γ is not a square in F when F is imaginary, and γ is not totally negative when F is real. The proof of this depends in part upon a theorem of Hilbert on the existence of relative units in a field over F. Aside from this theorem the entire treatment is elementary. *H. W. Brinkmann.*

Mahler, Kurt. Remarks on ternary Diophantine equations. Amer. Math. Monthly 49, 372–378 (1942). [MF 6792]

In a previous paper [Math. Ann. 107, 691–730 (1933)], the author obtained the following generalization of Thue's theorem: If the greatest prime factor of a binary form $F(x, y)$ is bounded for an infinite number of relatively prime pairs x, y, then $F(x, y)$ is a power of a linear or quadratic form. He proves now by simple examples that the assertions of Thue's theorem and of his generalization are no longer true if the binary form $F(x, y)$ is replaced by a ternary form $F(x, y, z)$. *C. L. Siegel* (Princeton, N. J.).

Pall, Gordon. Quaternions and sums of three squares. Amer. J. Math. 64, 503–513 (1942). [MF 6945]

In the present paper, employing the results of his own investigations [Trans. Amer. Math. Soc. 47, 487–500 (1940); these Rev. 2, 36; Ann. of Math. (2) 41, 754–766 (1940); these Rev. 2, 36] and those of C. L. Siegel [Acta Arith. 1, 83–86 (1935)], the author proves the crucial result of the first part of a paper by U. V. Linnik [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 363–402 (1940); these Rev. 2, 348], namely, that every integral quaternion t of norm p, p an odd prime, is a right-divisor of at least one of the quaternions x_0+x, where $x=i_1x_1+i_2x_2+i_3x_3$ is one of the $r(m)$ pure and proper integral quaternions of norm m and $x_0^2\equiv -m \pmod p$, provided $m\neq 4k$ or $8k+7$ and m is sufficiently large. Linnik's original proof of this statement contained an erroneous estimate of the number of representations of a binary form as a sum of three squares. *A. E. Ross* (St. Louis, Mo.).

Vandiver, H. S. An arithmetical theory of the Bernoulli numbers. Trans. Amer. Math. Soc. 51, 502–531 (1942). [MF 6551]

This paper discusses in detail for generalizations of Bernoulli numbers certain properties allied to the famous formula of von Staudt-Clausen. Proofs are given of theorems announced previously [Proc. Nat. Acad. Sci. U. S. A. 23, 555–559 (1937)] as well as later results. Throughout the paper the underlying method of obtaining equalities is the following: If for every prime p (or at least infinitely many primes) we have the congruence $a\equiv b \pmod p$ and if a and b do not depend on p, then $a=b$. The method consists in setting up identities in one or more indeterminates involving the functions $(x^m-1)/(x-1)$, operating n times by $\theta=xd/dx$, reducing the result to a congruence modulo p and applying the above principle.

The general Bernoulli number considered is written $b_n(m_r, m_{r-1}, \cdots, m_0)$ and depends on r parameters m_i which are nonzero integers. It is defined by

$$b_n(m_r, \cdots, m_0)=(m_rb^{(r)}+m_{r-1}b^{(r-1)}+\cdots+m_1b^1+m_0)^n,$$

in which the right member is expanded in full by the multinomial theorem, after which all superscripts are abandoned and exponents of b are degraded to subscripts; then b_k is taken to mean the kth Bernoulli number ($b_0=1$, $b_1=-1/2$, $b_2=1/6$, $\cdots$). The ultra-Bernoulli number $b_n(1, 1, \cdots, 1)$ of Lucas is thus a special case. Much of the paper is devoted to a detailed discussion of the case $r=2$, whose properties are more interesting than the more general Bernoulli numbers. An interesting theorem of the von Staudt-Clausen type states that, for n even,

$$b_n(k, h, j)=I/jk+(n-1)\sum 1/p_i,$$

where I is an integer, and the p's are the "von Staudt-Clausen primes of order n," that is, primes p such that n is divisible by $p-1$. Hence the prime factors of the denominators of Bernoulli numbers of the second order are (except for the prime factors of j and k) the same as those of the first order. For Bernoulli numbers of orders not less than 3 other primes appear as factors of the denominators. *D. H. Lehmer* (Berkeley, Calif.).

Vandiver, H. S. General congruences involving the Bernoulli numbers. Proc. Nat. Acad. Sci. U. S. A. 28, 324–328 (1942). [MF 7067]

Congruences of the Kummer type for ordinary Bernoulli numbers are generalized to the Bernoulli numbers of the rth order discussed by the author in a recent paper [see the

preceding review]. The results are deduced from a theorem on differentiation in rings of polynomials given by the author [Proc. Nat. Acad. Sci. U. S. A. 28, 24–27 (1942); these Rev. 3, 265, where in the second display $(f_{n_1}+\cdots+f_{n_s})$ should be replaced by $(f_{n_1}+\cdots+f_{n_s})'$]. The extension of the concepts of regular and irregular primes and of proper and improper divisors of Bernoulli numbers to Bernoulli numbers of the second order is likewise discussed.

D. H. Lehmer (Berkeley, Calif.).

Carlitz, L. Some topics in the arithmetic of polynomials. Bull. Amer. Math. Soc. 48, 679–691 (1942). [MF 7266]

In this paper, which was an invited address before the American Mathematical Society, the author reviews several topics in the arithmetic of polynomials in a finite Galois field. This arithmetic is similar to ordinary arithmetic and, in a good many respects, is simpler. Thus the ζ-function for this arithmetic is an elementary function and certain arithmetical functions have correspondingly simple properties. The author discusses these briefly and then turns to the discussion of the representation of a polynomial by sums of squares. The number of such representations can be given by certain "divisor functions," which is not always possible in ordinary arithmetic. Next, he discusses various special polynomial functions that are of importance in the theory such as the "linear" function. Finally he defines analogues of the ordinary Bernoulli numbers; here the main result is an analogue of the von Staudt theorem in arithmetic.

H. W. Brinkmann (Swarthmore, Pa.).

James, R. D. and Weyl, H. Elementary note on prime number problems of Vinogradoff's type. Amer. J. Math. 64, 539–552 (1942). [MF 6948]

The object of this paper is to indicate how Vinogradoff's method of dealing with the problem of the number of representations of a large number N as a sum of three primes can be applied to the quite general problem described below. An attempt is made to produce actual asymptotic formulas, although these are not very explicit because of the generality of the problem. Let

$$A(N,k)=A_r\left(N,k;\begin{matrix}a_1,\cdots,a_r\\ \beta_1,\cdots,\beta_r\end{matrix}\right)$$

denote the number of representations of an integer k in the form $k=a_1p_1+\cdots+a_rp_r$ by means of primes subject to the inequalities $p_1\leqq\beta_1N,\cdots,p_r\leqq\beta_rN$. Here the a's are nonzero integers, $r\geqq3$ and the β's are positive. The main result of the paper is that

$$A(N,k)=R(N,k)S(k)+(\rho/N)(N/\log N)^r,$$

where ρ tends to zero (uniformly in k as $N\to\infty$) more rapidly than any negative power of $\log N$. The "singular series" $S(k)$ does not depend on the β's and is given explicitly by

$$S(k)=S(k;a_1,\cdots,a_r)=\sum_{q=1}^{\infty}\phi(q)\frac{\mu(q/(q,k))}{\phi(q/(q,k))}\prod_{r=1}^{r}\frac{\mu(q/(q,a_r))}{\phi(q/(q,a_r))},$$

where (t,u) denotes the greatest common divisor of t and u and ϕ and μ are the arithmetical functions of Euler and Möbius. The function

$$R(N,x)=R\left(N,x;\begin{matrix}a_1,\cdots,a_r\\ \beta_1,\cdots,\beta_r\end{matrix}\right)$$

is given as the Fourier transform

$$R(N,x)=\int_{-\infty}^{\infty}e^{-2\pi ixz}\prod_{r=1}^{r}\int_{N^{\frac{1}{2}}}^{\beta_rN}e^{2\pi ia_rz^{-1}}(\log t)^{-1}dtdz.$$

A probability interpretation of $R(N,x)$ is given as follows. Let

$$\phi_r(x)=\begin{cases}1/\log x & N^{\frac{1}{2}}\leqq x\leqq\beta_rN,\\ 0 & \text{elsewhere}\end{cases}$$

($1/\log x$ being the density function for primes). Then, if the r primes p_r be replaced by r statistically independent continuous variables x_r such that the probability of x_r lying between x and $x+dx$ is given by $\phi_r(x)dx$, then $R(N,x)$ is the probability that $a_1x_1+\cdots+a_rx_r$ lies between x and $x+dx$.

The function $R(N,x)$ is shown to possess an asymptotic expansion

$$R(N,x)\sim\gamma_0(x)+\gamma_1(x)(\log N)^{-1}+\gamma_2(x)(\log N)^{-2}+\cdots.$$

The functions $\gamma_n(x)$ are not determined for $n>0$. For $\gamma_0(x)$ is given the expression

$$\gamma_0(x)=\{(r-1)!a_1a_2\cdots a_r\}^{-1}\Delta_1\Delta_2\cdots\Delta_rx^{r-1},$$

where the operator $\Delta_rf(x)=f(x)-f(x-a_r\beta_r)$. Two simple examples (with $r=3$) of $\gamma_0(x)$ are given. There is finally a brief general discussion of the lth power problem of the number of representations of any integer k in the form $k=a_1x_1^l+\cdots+a_rx_r^l$ with $x_r^l\leqq\beta_rN$ $(\nu=1,2,\cdots,r)$.

D. H. Lehmer (Berkeley, Calif.).

Behrend, F. A. On the frequency of the primes. J. Proc. Roy. Soc. New South Wales 75, 169–174 (1942). [MF 6965] (identical with the paper in 1939)

This paper is a sequel to an earlier paper [J. London Math. Soc. 15, 257–259 (1940); these Rev. 2, 249] in which it is proved by elementary properties of integers that the number $\pi(n)$ of primes not exceeding n satisfies the inequalities

$$\tfrac{1}{2}<\pi(n)\lambda(n)/n<12,\qquad n\geqq2.$$

Here the numerical function $\lambda(n)=\sum_1^n\nu^{-1}$ plays the role of the transcendental function $\log x$. The present paper sharpens the above inequalities to

$$\tfrac{1}{2}<\pi(n)\lambda(n)/n<3,\qquad n\geqq2.$$

[Compare T. S. Broderick's papers: J. London Math. Soc. 14, 303–310 (1939); these Rev. 1, 41; Proc. Roy. Irish Acad. Sect. A. 46, 17–24 (1940); these Rev. 1, 292.]

D. H. Lehmer (Berkeley, Calif.).

Wintner, Aurel. On an elementary analogue of the Riemann-Mangoldt formula. Bull. Amer. Math. Soc. 48, 759–762 (1942). [MF 7280]

In connection with Ramanujan's attempt to obtain the prime number theorem, Hardy [Ramanujan, Cambridge University Press, Cambridge, England, 1940, chap. 2; these Rev. 3, 71] gives an expansion for the function $\sum_{n=1}^{\infty}p^nx^{p^n}$ of the same type as the explicit formula of Riemann and von Mangoldt. The author shows that the expansion can be obtained in a simple way and that this approach was found by Dedekind, whose purpose was to show that the argument later used by Ramanujan is incorrect. The present proof of the expansion corrects two errors in the formula as it is given by Hardy.

H. S. Zuckerman.

Mann, Henry B. A proof of the fundamental theorem on the density of sums of sets of positive integers. Ann. of Math. (2) 43, 523–527 (1942). [MF 7006]

If A is a set of nonnegative integers a, $A(n)$ denotes the number of a's with $0<a\leqq n$. The sum C of two such sets A and B consists of all the numbers $a+b$, where $a\subset A$ and $b\subset B$. Let $0\subset A$, $0\subset B$. Let $A(x)\geqq\alpha x$, $B(x)\geqq\beta x$ for

$x=1, \cdots, n$. Then $C(n) \geqq n \cdot \min(1, \alpha+\beta)$. This theorem has been conjectured since 1930, but the most ingenious efforts yielded only weaker estimates or special cases. The author gives its first proof. He even proves the stronger theorem: If $0 \subset A$, $0 \subset B$, then

$$\frac{C(n)}{n} \geqq \min_{\substack{0<m\leqq n \\ m\notin C}} \left(1, \frac{A(m)+B(m)}{m}\right).$$

This refinement, as well as some simplifications of the original proof, are due to A. Brauer. A second theorem (in even stronger form) is implicitly contained in a paper by A. S. Besicovitch [J. London Math. Soc. 10, 246–248 (1935)]. *P. Scherk* (Bloomington, Ind.).

Erdös, P. On an elementary proof of some asymptotic formulas in the theory of partitions. Ann. of Math. (2) 43, 437–450 (1942). [MF 7001]

The main purpose of this paper is to give an elementary proof of the result

$$p(n) \sim (a/n)e^{c\sqrt{n}},$$

where $p(n)$ is the number of partitions of n, $c=\pi(2/3)^{\frac{1}{2}}$ and a is a constant. The author is unable to show by his methods that $a=48^{-\frac{1}{2}}$, the value found by Hardy and Ramanujan [Proc. London Math. Soc. (2) 17, 75–115 (1918)]. He states that his methods apply to the proof of a similar result for the number of partitions of n into integers congruent to one of a given set of r numbers (mod m). Various other applications of his methods are indicated.

B. W. Jones (Ithaca, N. Y.).

Heinhold, Josef. Zur Geometrie der Zahlen. Math. Z. 47, 199–214 (1941). [MF 6819]

Let $f(x, y)$ be the distance function of a convex domain and consider the lower bound $k=k(f)$ of all positive constants c, such that for arbitrary real numbers ξ, η the inequality $f(x+\xi, y+\eta) \leqq c$ has an integral solution x, y. It is proved that there exists a unimodular transformation $x=\alpha x'+\beta y'$, $y=\gamma x'+\delta y'$ and two real numbers u, v satisfying $0 \leqq u \leqq v \leqq \frac{1}{4}$ and $f(x_r, y_r)=k$ $(r=1, 2, 3)$, where $x_1'=\frac{1}{4}+u$, $y_1'=\frac{1}{4}+v$, $x_2'=\frac{3}{4}-u$, $y_2'=-\frac{1}{4}-v$, $x_3'=\frac{1}{4}+u$, $y_3'=-\frac{3}{4}+v$. The result is applied in the special case $f^p=|Ax+By|^p+|Cx+Dy|^p$ with real A, B, C, D, $AD-BC\neq 0$, $p \geqq 1$.

C. L. Siegel (Princeton, N. J.).

Tietze, Heinrich. Über die Anzahl komprimierter Gitterpunktmengen von gegebener Punktezahl. Math. Z. 47, 352–356 (1941). [MF 6722]

This note is concerned with the generalization to n dimensions of the Ferrers reduced graph of a partition considered also in a recent paper of the author [Abh. Math. Sem. Hansischen Univ. 14, 273–284 (1941); these Rev. 3, 166]. A finite set M of lattice points in n dimensional space is called reduced (komprimiert) if first all points have positive integer coordinates and, secondly, if $(x)=(x_1, x_2, \cdots, x_n)$ be any member of M then any point (x') whose coordinates do not exceed the corresponding coordinates of (x) also belongs to M. Let m be the number of points of M. Clearly if n and m are fixed there are only finitely many such reduced point sets M. The number of these is denoted by $P^n(m)$; $P^0(m)$ is taken as unity, which is also the value of $P^1(m)$. The function $P^2(m)$ is, of course, the celebrated partition function $p(m)$, the number of unrestricted partitions of m. By a dissection process, it is shown inductively that $P^n(m)$, for m fixed, is a polynomial in n of degree $m-1$. Hence if $P^k(m)$ is known for $k=0, 1, \cdots, m-1$, a formula for $P^n(m)$ for this fixed value of m can be computed. Formulas for $P^n(m)$ for $m=2, 3, 4$ have been given previously [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1940, 166]. In the present paper the functions for $m=5$ and 6 are found to be

$$4!P^n(5)=n^4+18n^3-n^2+6n,$$
$$5!P^n(6)=n^5+40n^4-40n^2+24n.$$

D. H. Lehmer (Berkeley, Calif.).

Pillai, S. S. On algebraic irrationals. Proc. Indian Acad. Sci., Sect. A. 15, 173–176 (1942). [MF 7041]

One of the results of the paper reviewed below is given with slight alterations in the proof. *H. S. Zuckerman.*

Pillai, S. S. On a problem in Diophantine approximation. Proc. Indian Acad. Sci., Sect. A. 15, 177–189 (1942). [MF 7042]

Questions concerning the order of $S(\theta, N)=\sum_{n=1}^{N}\{n\theta\}$, with $\{x\}=x-[x]-\frac{1}{2}$, have been discussed by Hardy and Littlewood [Proc. London Math. Soc. (2) 20, 15–36 (1921)] and by others. In this paper a reduction formula is obtained. This expresses $S(\theta, N)$ by means of $S(\theta, N_1)$, where N_1 is decidedly smaller than N. By suitable applications of the reduction formula the author proves the results of Hardy and Littlewood and some further results. The proofs are all elementary in character, depending upon properties of the continued fraction expansion of θ, while Hardy and Littlewood, besides using the continued fraction, also make use of transcendental methods. The proofs of theorems VII and IX are not quite complete but the details are easily supplied if the condition $n>7$ is added to theorem VII. Most of the misprints are not serious but an inequality is inverted in theorem XVI. The inequality in the conclusion should read $S(\theta, N)<\delta$. *H. S. Zuckerman.*

ANALYSIS

Fourier Series and Generalizations, Integral Transforms

Hardy, G. H. Note on Lebesgue's constants in the theory of Fourier series. J. London Math. Soc. 17, 4–13 (1942). [MF 6968]

The author gives two new formulas for the Lebesgue constant

$$L_n=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{|\sin Nt|}{\sin t}dt,$$

where $N=2n+1$, namely,

$$L_n=\int_0^{\infty}\frac{\tanh Ny}{\tanh y}\frac{dy}{y^2+\frac{1}{4}\pi^2},$$

$$L_n=\frac{4}{\pi^2}\int_0^{\infty}\frac{\sinh Ny}{\sinh y}\log\coth\tfrac{1}{2}Ny\,dy.$$

From these he deduces for L_n the formula of Szegö [Math.

Z. 9, 163–166 (1921)] and the asymptotic series of Watson [Quart. J. Math., Oxford Ser. 1, 310–318 (1930)].

A. Zygmund (South Hadley, Mass.).

Kiang, Süe-yung. Über die Fouriersche Entwicklung der singulären Funktion bei einer Lebesgueschen Zerlegung. Math. Z. 47, 330–342 (1941). [MF 6720]

Let

$$f(x)=a_0/2+\sum_1^\infty (a_n\cos nx+b_n\sin nx)$$

be the Fourier series of an increasing bounded function in $0\leqq x\leqq 2\pi$ ($f(0)=0$), and let $f(x)=T(x)+S(x)$ be the Lebesgue decomposition of $f(x)$, where $T(x)$ is absolutely continuous and $S(x)$ the "function of the singularities." The purpose of this paper is to establish a limiting process permitting the calculation of $S(2\pi)$ (and more generally of $S(x)$) as a function of the coefficients a_n, b_n. *R. Salem.*

Szász, Otto. On the logarithmic means of rearranged partial sums of a Fourier series. Bull. Amer. Math. Soc. 48, 705–711 (1942). [MF 7268]

Let $\frac{1}{2}a_0+\sum_1^\infty a_n\cos n\theta$ be the Fourier series (S) of the even function $f(\theta)$ belonging to the class L, and let S_r be the rth partial sum of (S) for $\theta=0$. Hardy and Littlewood [Proc. Cambridge Philos. Soc. 31, 317–323 (1935)] have proved that, if (1) $f(\theta)=o(|\log\theta|^{-1})$ as $\theta\to 0$, then

$$\sum_0^n \frac{S_\nu^*}{\nu+1}=o(\log n), \tag{2}$$

where $S_0^*, S_1^*, \cdots, S_n^*$ are the numbers $|S_0|, |S_1|, \cdots, |S_n|$ rearranged in decreasing order. The relation (2) has many interesting consequences and the question of whether (2) holds when (1) is replaced by

$$\int_0^\theta |f(t)|\,dt=o(\theta|\log\theta|^{-1}) \tag{3}$$

is open. The main result of the present paper, which follows Hardy and Littlewood's argument, is that the relation (2) is a consequence of (3) under the additional hypothesis $|f(\theta)|<\theta^{-c}$ as $\theta\to 0$, c being any positive number.

R. Salem (Cambridge, Mass.).

Szász, Otto. On the partial sums of harmonic developments and of power series. Trans. Amer. Math. Soc. 52, 12–21 (1942). [MF 6991]

Let $H(r,\theta)=\sum_1^\infty b_\nu r^\nu\sin\nu\theta$ be a harmonic sine development convergent for $0<r<1$ and nonnegative for $0<\theta<\pi$, and let $s_n(r,\theta)=\sum_1^n b_\nu r^\nu\sin\nu\theta$. There exists a number R_n depending on n only and such that $s_n(r,\theta)\geqq 0$ for $0<r\leqq R_n$, but not necessarily for $r>R_n$. The author investigates the properties of the numbers R_n and, in particular, proves that (i) R_n is the largest r such that $\sum_1^n \nu r^\nu\sin\nu\theta\geqq 0$ for $0<\theta<\pi$; (ii) $R_n=1-3n^{-1}\log n+n^{-1}(\log\log n+o(1))$; (iii) if $f(\theta)\sim\sum_1^\infty b_\nu\sin\nu\theta$ is positive and convex upwards in $0<\theta<\pi$, then $\sum_1^n r^\nu b_\nu\sin\nu\theta$ is convex upwards in $0<\theta<\pi$ for $0<r\leqq R_n$, but not always for $r>R_n$; (iv) if $g(\theta)\sim\frac{1}{2}a_0+\sum_1^\infty a_\nu\cos\nu\theta$ is increasing in $0<\theta<\pi$, so is the partial sum $\frac{1}{2}a_0+\sum_1^n a_\nu r^\nu\cos\nu\theta$ for $0<r\leqq R_n$, and here again R_n cannot be replaced by anything larger. *A. Zygmund* (South Hadley, Mass.).

Szász, Otto. On convergence and summability of trigonometric series. Amer. J. Math. 64, 575–591 (1942). [MF 6950]

The series $\sum_1^\infty a_n$ is said to be summable L (Lebesgue's method) to the value s if the series $\sum_1^\infty n^{-1}a_n\sin nt=F(t)$ converges in an interval $(0,t_0)$ and if $t^{-1}F(t)\to s$ when $t\to+0$. The purpose of the paper is to generalize some results on L summability due to Hardy and Littlewood. Among several theorems: (1) If, when $\lambda\to 1+0$,

$$\liminf_{n\to\infty}\ \min_{n<k\leqq\lambda n}\ \sum_n^k a_\nu\geqq 0,$$

L summability to s involves ordinary convergence to s. (2) If $\sum_n^{2n}(|a_\nu|-a_\nu)=O(1)$ for $n\to\infty$, ordinary convergence implies L summability. (3) If $\sum_n^{2n}(|a_\nu|-a_\nu)=O(n^{1-\delta})$ for every positive $\delta<1$ and if $a_1+a_2+\cdots+a_n-s=O(1/\log n)$ for $n\to\infty$, then the series is summable L to s.

R. Salem (Cambridge, Mass.).

Wang, Fu Traing. The absolute Cesàro summability of trigonometrical series. Duke Math. J. 9, 567–572 (1942). [MF 7336]

If the series $\sum(a_n^2+b_n^2)(\log n)^{1+\epsilon}$ ($\epsilon>0$) is convergent, the series $\sum(a_n\cos nx+b_n\sin nx)$ is summable $|C,\alpha|$ almost everywhere for any $\alpha>\frac{1}{2}$. [Cf. also the author's previous paper, J. London Math. Soc. 16, 174–176 (1941); these Rev. 3, 231.] *A. Zygmund* (South Hadley, Mass.).

Wang, F. T. On Riesz summability of Fourier series. Proc. London Math. Soc. (2) 47, 308–325 (1942). [MF 6598]

Wang, Fu Traing. On Riesz summability of Fourier series. II. J. London Math. Soc. 17, 98–107 (1942). [MF 7317]

In the first paper, the author studies the summability (R,λ,k) of Fourier-Lebesgue series. The greater part of the paper is concerned with the case when $\lambda(\omega)=\exp\omega^\alpha$, $0<\alpha<1$. The following theorem may serve as sample of the results obtained. (1) If

$$\int_0^t |f(x+u)+f(x-u)-2f(x)|\,du=o(t) \tag{*}$$

for some x, and if at the point x the Fourier series of the function f is summable $(R,\exp\omega^\alpha,k)$ for some k, then it is also summable $(R,\exp\omega^\alpha,h)$ for any $h>0$. In addition, the author completes a result of Hardy and Littlewood who showed [Ann. Scuola Norm. Super. Pisa (2) 3, 43–46 (1934)] that, if the integral in (*) is $o(t|\log t|^{-1})$, and if the Fourier coefficients a_n, b_n of f are $O(n^{-\delta})$, $\delta>0$, then the Fourier series of f converges at the point x to the value $f(x)$. They left open the question raised by themselves as to whether the integral condition may be replaced by the less stringent one

$$\int_0^t \{f(x+u)+f(x-u)-2f(x)\}\,du=o(t|\log t|^{-1}). \tag{**}$$

The author answers this question in the negative, even if a_n and b_n are $O(n^{-(1-\epsilon)})$, $\epsilon>0$. He also shows that (**) is sufficient provided that a_n and b_n are $O(n^{-1}\log n)$. This result is obtained as an application of the summability $(R,\exp\omega^\alpha,k)$. The second part of the paper contains results concerning convergence and summability $(R,\exp\omega^\alpha,k)$ of Fourier series under conditions concerning the behavior of higher integrals of f. *A. Zygmund.*

Chow, Hung Ching. On the summability factors of Fourier series. J. London Math. Soc. 16, 215–220 (1941). [MF 6616]

Let $\sum(a_n\cos nx+b_n\sin nx)$ be the Fourier series of a function $f(x)$ of class L. (1) If $\{\lambda_n\}$ is a convex sequence and

$\sum n^{-1}\lambda_n$ converges, the series (*) $\sum\lambda_n(a_n\cos nx+b_n\sin nx)$ is summable $|C,1|$ for almost all x. (2) If $f(x)$ and its conjugate both belong to class L, and if $\{\lambda_n\}$ is such that the two series $\sum n(\lambda_n-\lambda_{n+1})^2$ and $\sum n^{-1}\lambda_n^2$ converge, the series (*) is summable $|C,1|$ for almost all x; if $\{\lambda_n\}$ is convex it is sufficient to assume the convergence of $\sum n^{-1}\lambda_n^2$.

R. Salem (Cambridge, Mass.).

Chow, Hung Ching. On the absolute summability of Fourier series. J. London Math. Soc. 17, 17–23 (1942). [MF 6970]

Let

$$\tfrac{1}{2}a_0+\sum_1^\infty a_n\cos nx+b_n\sin nx$$

be the Fourier series (S) of $f(x)$ of period 2π. If $f(x)$ belongs to Lip (k,p), that is to say, if

$$\int_{-\pi}^{+\pi}|f(x+t)-f(x)|^p dx=O(|t|^{kp}),$$

and if $1<p\leqq 2$, $0<k\leqq 1$, $kp>1$, then by a theorem of Hardy and Littlewood (S) is absolutely convergent. The purpose of the paper is to extend this theorem by proving that, under the same hypothesis, the series (S) is absolutely summable $|C,\alpha|$ for $\alpha>1/p-k$.

R. Salem.

Salem, R. On sets of multiplicity for trigonometrical series. Amer. J. Math. 64, 531–538 (1942). [MF 6947]

Let P be a perfect nondense set constructed on the interval $I=[0,2\pi]$ by the familiar Cantor procedure. We divide I into three parts of lengths proportional, respectively, to the numbers ξ_1, $1-2\xi_1$, ξ_1, where $0<\xi_1<\frac{1}{2}$. We remove the interior of the central part, and each of the remaining parts we divide into three parts of lengths proportional to ξ_2, $1-2\xi_2$, ξ_2, where $0<\xi_2<\frac{1}{2}$. We again remove the interiors of the central parts, and proceed in this way. In the limit we get the set P. Let us now consider the family Π of all the sets P for which $a_k\leqq\xi_k\leqq b_k$ with $b_k-a_k\geqq 1/\omega(k)$ and $\log\omega(k)=o(k)$ (the latter condition implies that the sequences $\{a_k\}$ and $\{b_k\}$ should not be too close to each other). It is shown that, if

$$(*)\qquad \liminf(a_1a_2\cdots a_k)^{1/k}=\alpha>0,$$

then "almost every" set P from Π is a set of multiplicity for trigonometric series. In other words, there exists then a nondecreasing function $F(x)$ constant in the intervals adjacent to P and having Fourier-Stieltjes coefficients c_n tending to 0. It is even shown that under the condition (*) for "almost all" sets P from Π we have $c_n=o(n^{-\delta})$, $\delta>0$. [The existence of functions F with this property was first shown by Littlewood, Quart. J. Math., Oxford Ser. 7, 219–226 (1936)]. "Almost all" in the above theorems has the following meaning. We set $\xi_k=(b_k-a_k)\eta_k+a_k$, so that $0\leqq\eta_k\leqq 1$. Then, for almost all (in the familiar sense) sequences $\{\eta_k\}$, the sets P have the required properties.

A. Zygmund (South Hadley, Mass.).

Salem, R. On singular monotonic functions of the Cantor type. J. Math. Phys. Mass. Inst. Tech. 21, 69–82 (1942). [MF 7248]

A function $F(x)$ defined in the interval $(0,2\pi)$ is said to be of type C (C for Cantor) if it is nondecreasing and is constant in each interval contiguous to a perfect set of measure 0. Littlewood proved [Quart. J. Math., Oxford Ser. 7, 219–226 (1936)] that there exist functions $F(x)$ of type C and such that

$$(*)\qquad \int_0^{2\pi}e^{inx}dF(x)=O(n^{-\alpha})$$

for some $\alpha>0$. Wiener and Wintner [Amer. J. Math. 60, 513–522 (1938); J. Math. Phys. Mass. Inst. Tech. 17, 233–246 (1939)] proved that for every $\epsilon>0$, however small, there exists a nondecreasing function $F(x)$ satisfying (*) with $\alpha=\frac{1}{2}-\epsilon$. There $F(x)$ is, however, not of type C, and in the present paper Salem shows that there exist functions of type C satisfying (*) with $\alpha=\frac{1}{2}-\epsilon$. This extension is important for the problem of uniqueness of trigonometric series, for it shows the existence of trigonometric series $\frac{1}{2}a_0+\sum_{n=0}^\infty(a_n\cos nx+b_n\sin nx)$ converging to 0 almost everywhere, but not everywhere, and having coefficients $O(n^{-\frac{1}{2}+\epsilon})$, a result which seems not to be deducible from the Wiener-Wintner construction. The paper also contains proofs of the following facts: (a) A one-one mapping $y=ax^2+bx+c$ of the interval $(0,2\pi)$ into itself may transform a set of uniqueness into one of multiplicity [it is known that a linear transformation preserves the character of uniqueness of a point set [see J. Marcinkiewicz and A. Zygmund, Rec. Math. [Mat. Sbornik] N.S. 2 (44), 733–737 (1937)]]; (b) if $F(x)$ is of bounded variation and continuous, the sequence of the indices n such that $\int_0^{2\pi}e^{inx}dF(x)$ does not tend to 0 may, in a certain sense, be as "sparse" as we please [that this sequence is of density 0 is a very well-known result of Wiener [see J. Math. Phys. Mass. Inst. Tech. 3, 72–94 (1924)]].

A. Zygmund.

Herriot, John G. Nörlund summability of double Fourier series. Trans. Amer. Math. Soc. 52, 72–94 (1942). [MF 6995]

The author studies the behavior of the Nörlund means

$$(*)\qquad P_n^{-1}\sum_{k=0}^n p_{n-k}s_{kk}(x,y;f),\quad P_n=p_0+p_1+\cdots+p_n,$$

for double Fourier series. Here $s_{mn}(x,y;f)$ denotes the (rectangular) partial sums of the Fourier series of a function $f(x,y)$ integrable over the square $0\leqq x\leqq 2\pi$, $0\leqq y\leqq 2\pi$. It is shown that, if the Nörlund method defined by the sequence $\{p_k\}$ satisfies, besides the conditions of regularity, certain additional conditions, then the sequence (*) converges to $f(x,y)$ almost everywhere. The additional conditions are not quite simple and they are omitted here, but they are satisfied by the method (C,α), $\alpha>0$, so that the sequence $s_{nn}(x,y)$ is summable (C,α), $\alpha>0$, to $f(x,y)$ at almost every point (x,y). In the case $\alpha=1$ the theorem was also proved by Grünwald [Acta Univ. Szeged. Sect. Sci. Math. 10, 55–63 (1941); these Rev. 2, 280] but Herriot's result was obtained independently from Grünwald's.

Furthermore, the paper discusses the behavior of the double Nörlund means

$$(**)\qquad (P_m^{(1)}P_n^{(1)})^{-1}\sum_{k=0}^m\sum_{l=0}^n p_{m-k}^{(1)}p_{n-l}^{(2)}s_{kl}(x,y;f),$$

$$P_m^{(i)}=p_0^{(i)}+p_1^{(i)}+\cdots+p_m^{(i)},$$

of the Fourier series of $f(x,y)$. It is shown that if the numbers $p_k^{(i)}$ satisfy certain conditions then the means (**) tend to $f(x,y)$ almost everywhere, provided that the ratios m/n and n/m are bounded. In the case when the means (**) are the (C,α,β) means, with $\alpha>0$, $\beta>0$, this theorem reduces to a theorem of Marcinkiewicz and the reviewer [Fund. Math. 32, 112–132 (1939)].

The author also discusses the localization property of the means (*). He proves that from certain general theorems it follows in particular that, if $f(x, y)$ vanishes in a neighborhood of a point (x_0, y_0), then at that point the (C, α) means of the sequence $s_{mn}(x_0, y_0; f)$ tend to 0 for $\alpha \geqq 1$, but not necessarily for $0<\alpha<1$. Similar results are also obtained for the means (**). *A. Zygmund* (South Hadley, Mass.).

Young, L. C. On the convergence of Fourier-Bessel series. Proc. London Math. Soc. (2) 47, 290–307 (1942). [MF 6597]

Previous results on the convergence of the Fourier-Bessel expansion of a function in the range $0 \leqq x \leqq 1$ are extended by weakening the conditions imposed on the behavior of the function at the end points of the interval (0, 1). The principal theorems proved are: (1) if $x^{\frac{1}{2}}f(x)$ is a function of bounded variation in $0 \leqq x \leqq 1$, vanishing at $x=0, 1$, and if $x^{\frac{1}{2}}S_n(x)$ is the sum to n terms of its Fourier-Bessel expansion, then, as $n \to \infty$, $x^{\frac{1}{2}}S_n(x) \to \frac{1}{2}x^{\frac{1}{2}}[f(x+0)+f(x-0)]$ boundedly in $0<x<1$; further, if $x^{\frac{1}{2}}f(x)$ is continuous, $x^{\frac{1}{2}}S_n(x) \to x^{\frac{1}{2}}f(x)$ uniformly in $0 \leqq x \leqq 1$. (2) In (1) the term "function of bounded variation" may be interpreted to mean "function of bounded pth power variation" or "function of bounded Φ-variation," where $\Phi(u)=\exp(-u^{-c})$, $0<c<\frac{1}{2}$. The total Φ-variation of a function f in (0, 1) is defined as the upper bound of the sum $\sum_e \Phi(|\Delta f|)$ over any set e of nonoverlapping intervals Δ in (0, 1), where Δf is the difference of the values of f at the ends of Δ. *M. C. Gray.*

Hartman, Philip. The divergence of non-harmonic gap series. Duke Math. J. 9, 404–405 (1942). [MF 6877]

The author shows that for $\sum_{k=0}^{\infty}|a_k|^2=\infty$ and $\lambda_k/\lambda_{k-1} > q>1$ the series $\sum_{k=0}^{\infty}a_k e^{i\lambda_k t}$ is divergent for almost all t in $(-\infty, \infty)$. The assertion was known in the periodic case $\lambda_k=2\pi k$, but for nonarithmetic λ_k it had been known only for $q>\frac{1}{2}(5^{\frac{1}{2}}+1)$. The proof is very interesting. *S. Bochner* (Princeton, N. J.).

Peebles, G. H. On equivalence of certain types of series of orthonormal functions. Bull. Amer. Math. Soc. 48, 556–561 (1942). [MF 7048]

Let $\{u_n(x)\}$ be a given set of linearly independent functions, $\{v_n(x)\}$ and $\{w_n(x)\}$ two orthogonal sets formed from the u's on the interval (a, b) with the weight functions $p(x)$ and $q(x)=r(x)p(x)$. The equiconvergence of the expansions of an arbitrary function in terms of the $v_n(x)$ and $w_n(x)$ is studied. Sufficient conditions are obtained in terms of $r(x)$, especially in the case when $u_m(x)u_n(x)$ is a linear combination of the first $n+m+\beta$ members of the set $\{u_n(x)\}$ ($\beta=\beta(n, m)$ but bounded). *G. Szegö.*

Kosambi, D. D. On the zeros and closure of orthogonal functions. J. Indian Math. Soc. 6, 16–24 (1942). [MF 7153]

The author's principal result concerns a set $\{\varphi_n\}_1^{\infty}$ of uniformly bounded continuous orthonormal functions on (0, 1), which is "strongly random," that is, has the properties (i) $\liminf_{n\to\infty}\int_a^b \varphi_n^2(x)dx \geqq k(b-a)$, $k>0$, for all subintervals (a, b) of (0, 1), and (ii) $\max|\varphi_n| \geqq K>0$ between any two consecutive maxima of $\varphi_n(x)$, for sufficiently large n. The theorem is that, if such a set is closed with respect to continuous functions, then, for an infinite sequence of integers n, $\varphi_n(x)$ has not more than $n-1$ changes of sign in (0, 1). *E. S. Pondiczery* (Princeton, N. J.).

***Schmidli, Salomon. Über gewisse Interpolationsreihen.** Dissertation, Eidgenössische Technische Hochschule in Zürich, 1942. 70 pp.

In the first part of this thesis the author develops the properties of the "indicator" of a sequence $\{P_n(z)\}$ of polynomials [introduced by Pólya, Proc. London Math. Soc. (2) 24, 44–45 (1926)]. The indicator $p(\varphi)$ is defined as the greatest lower bound of numbers B for which the sequence $\{|P_n(z)|e^{-B|z|}\}$ is uniformly bounded on the ray $\arg z=\varphi$. It is shown that $p(\varphi)$, if finite, is the supporting function of a bounded convex region. Furthermore, the region obtained by reflecting the convex region in the real axis has the property that, outside any region whose boundary is at least ϵ (>0) outside it, the Borel transforms of the $P_n(z)$ are uniformly bounded; it is the smallest convex region with this property. In the second part a sufficient condition is given for the continuity of the Borel transform of an entire function $f(z)$ of exponential type on the boundary of its conjugate indicator diagram. This is

$$|f(re^{i\varphi})| \leqq S(\epsilon)r^{-1-\epsilon}e^{rh(\varphi)}, \quad r \geqq 0;\ 0 \leqq \varphi < 2\pi,$$

for every positive ϵ, where $S(\epsilon)$ depends on ϵ and $h(\varphi)$ is the indicator function of $f(z)$. In the third part these results are applied to a discussion of Abel's interpolation series

$$(1) \qquad f(z)=f(0)+\sum_{n=1}^{\infty}(z(z-n)^{n-1}/n!)f^{(n)}(n).$$

It is proved that a necessary condition for $f(z)$ to be developable in such a series (which necessarily converges uniformly in every finite region if it converges at one point other than $z=0$) is that $f(z)$ is entire and such that, uniformly in φ,

$$(2) \qquad f(re^{i\varphi})=o(r^2e^{rh(\varphi)}),$$

while a sufficient condition is (uniformly in φ)

$$(3) \qquad f(re^{i\varphi})=O(r^{-1-\epsilon}e^{rh(\varphi)})$$

for each positive ϵ; in (2) and (3), $h(\varphi)$ denotes the supporting function of the convex region defined by $|ze^{z+1}| \leqq 1$, $|z| \leqq 1$. These results are sharper than conditions previously given by various authors. The idea of the proof of the sufficient condition is to develop e^{sz}, as a function of z, into an Abel series, and to substitute the series into the representation of $f(z)$ by a contour integral, using the result of part II. Conditions are also given for the convergence of (1) absolutely, or as rapidly as a geometric series. In part IV, the same methods are applied to Stirling's series, but the results obtained are (as the author points out) less precise than those of Nörlund [Differenzenrechnung, Springer, Berlin, 1924, pp. 208–219]. In part V, the problem is to represent a function $f(z)$ by a series

$$\sum_{m=0}^{\infty}\sum_{j=0}^{p-1}F^{(mp+n_j)}(a_j)P_{mp+j}(z),$$

with suitable restrictions on the points a_j and integers n_j, where the $P_k(z)$ are polynomials. The author gives both sufficient conditions and necessary conditions for more general cases than those previously discussed by Poritsky, Gontcharoff and others [reference should also be made to Gelfond, Rec. Math. [Mat. Sbornik] N.S. 4 (46), 115–147 (1938)]. In particular, for the Lidstone series [$n_0=n_1=a_1=0$, $a_0=1$] the author gives as a necessary condition

$$|f(z)| < K(\epsilon)e^{(\pi+\epsilon)|z|}$$

for every positive ϵ, as a sufficient condition

$$|f(z)| < A|z|^{-1-\epsilon}e^{\pi|z|}$$

for some positive A and ϵ [for a necessary and sufficient condition of a different kind, see Widder, Trans. Amer. Math. Soc. 51, 387–398 (1942); these Rev. 3, 293].

E. S. Pondiczery (Princeton, N. J.).

Pi Calleja, Pedro. Note sur les intégrales singulières et leur application à la forme complexe de l'intégrale de Fourier. Bull. Soc. Math. France 68, 1–10 (1940). [MF 6805]

Extending the classical work of Lebesgue on singular integrals the author proves the following main result: In order that $I_n(f)\equiv\int_a^b f(x)\phi(x, n)dx$ converge (as $n\to\infty$) for every function $f(x)$ satisfying a Lipschitz condition at a point c interior to (a, b) it is necessary and sufficient that (i) for every $s>0$ there exists a positive number M_s, not depending on n, such that $\int_a^b|x-c|^s|\phi(x, n)|dx<M_s$; (ii) the integral $\int_a^b\phi(x, n)dx$ converges as well as $\int_a^b(x-\lambda)\phi(x, n)dx$ for every λ in (a, b).

J. D. Tamarkin.

Rey Pastor, J. Riemann's formula and the Laplace transformation. Univ. Nac. Tucumán. Revista A. 2, 217–243 (1941). (Spanish) [MF 6761]

Extending the work of various previous authors [Pincherle, Fujiwara, Tamarkin, Doetsch] the author considers the class of analytic functions for which the "Riemann integral"

$$\alpha(t)=(2\pi i)^{-1}\lim_{y\to\infty}\int_{k-iy}^{k+iy}f(u)e^{tu}du$$

converges uniformly in $t(0\leqq t<\infty)$ and shows that the following conditions are necessary and sufficient in order that $f(x)$ could be represented by $\int_0^\infty e^{-st}\alpha(t)dt$: (a) $f(z)$ is regular in a half-plane $x>c$; (b) $f(z)\to 0$ as $x\to\infty$, uniformly with respect to y; (c) $f(z)=o(|y|)$ as $|y|\to\infty$, in a half-plane $x>k>c$,

(d) $$\lim_{t\to\infty}e^{-ts}\int_{k-i\omega}^{k+i\omega}\frac{f(u)e^{tu}}{z-u}du=0.$$

In the rest of the paper the author discusses applications to the inversion theory of Laplace integrals and extensions to Laplace-Stieltjes integrals and to the case of "angular Riemann integrals," when the rectilinear path of integration is replaced by two sides of a sector with vortex at k.

J. D. Tamarkin (Providence, R. I.).

Ríos, Sixto. On the singularities of the Laplace integral. Portugaliae Math. 3, 110–114 (1942). (Spanish) [7084]

G. Doetsch [Theorie und Anwendung der Laplace Transformation, Julius Springer, Berlin, 1937] gave an example of a Laplace integral representing an analytic function having no singularities on the axis of convergence. The author shows that one of his theorems [Revista Mat. Hisp.-Amer. (2) 2, 26–29 (1936)] allows the construction of classes of Laplace-Stieltjes integrals having the same properties.

S. Mandelbrojt (Houston, Tex.).

Kober, H. A note on Hilbert's operator. Bull. Amer. Math. Soc. 48, 421–427 (1942). [MF 6707]

The operator $\mathfrak{H}$ considered carries a function $f(x)$ defined on $(-\infty, \infty)$ into

$$\mathfrak{H}f=\frac{1}{\pi}\int_{0+}^{\infty}\frac{f(x+t)-f(x-t)}{t}dt.$$

It is known that, when $f\in L_1(-\infty, \infty)$, $\mathfrak{H}f$ does not necessarily belong to $L_1(-\infty, \infty)$. The author shows that a necessary and sufficient condition for both f and $\mathfrak{H}f$ to belong to $L_1(-\infty, \infty)$ is that $f(t)=F(t)+G(t)$, where $F(t)$ and $G(t)$ are limit functions of functions of class $\mathfrak{H}_1$ and $\mathfrak{K}_1$, respectively [$\mathfrak{H}_1$ is the class of functions $F(x+iy)$ regular in $y>0$ and satisfying $\int_{-\infty}^{\infty}|F(x+iy)|dx\leqq M$; $\mathfrak{K}_1$ is the corresponding class for $y<0$; for the theory of classes $\mathfrak{H}_p$, which is used extensively by the author, see Hille and Tamarkin, Fund. Math. 25, 329–352 (1935)]. Let E denote the class of functions f satisfying this condition. With domain E, $\mathfrak{H}$ is a distributive closed unbounded transformation in $L_1(-\infty, \infty)$. E is unclosed and nowhere dense in $L_1(-\infty, \infty)$; its closure is the set of elements $f(t)$ of $L_1(-\infty, \infty)$ with $\int_{-\infty}^{\infty}f(t)dt=0$. This last condition is not sufficient for f of L_1 to have $\mathfrak{H}f\in L_1$ [example due to H. R. Pitt]. The theory is illustrated with several examples.

R. P. Boas, Jr.

Varma, R. S. A self-reciprocal function. Duke Math. J. 9, 259–261 (1942). [MF 6866]

The author claims that the function

$$f(x)=x^{\nu-\mu-\sigma+\frac{1}{2}}e^{-\frac{1}{2}x^2}J_\mu(x)J_\sigma(x)$$

is self-reciprocal in the Hankel transform of order ν. The reviewer finds, however, that the proof is invalid, and does not think that there is any self-reciprocal function of the form $f(x)$.

M. C. Gray (New York, N. Y.).

Mohan, B. A pair of self-reciprocal functions. Proc. Benares Math. Soc. (N.S.) 2, 39–42 (1940). [MF 6646]

If n is a positive integer and (k, r) denotes $k(k+1)\cdots(k+r-1)$, the function

$$x^{\lambda-2n+\frac{1}{2}}e^{-\frac{1}{2}x^2}\sum_{r=0}^{\infty}((-n, r)/r!)\ {}_1F_1(-r; 1-n+\tfrac{1}{2}\lambda+\tfrac{1}{2}\nu; \tfrac{1}{2}x^2),$$

and a similar one, are self-reciprocal in the Hankel transform of order λ. The derivation parallels that in another paper of the author [J. Indian Math. Soc. (N.S.) 5, 123–127 (1941); these Rev. 3, 238].

R. P. Boas, Jr.

Banerjee, D. P. On the properties of the functions which are self-reciprocal in Hankel's transform. Bull. Calcutta Math. Soc. 33, 93–97 (1941). [MF 7024]

If the function

$$f(x)=(i2\pi)^{-1}\int_{c-i\omega}^{c+i\omega}2^{\frac{1}{2}s}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s)\psi(s)x^{-s}ds,\ \psi(s)=\psi(1-s),$$

is R_μ and if a sequence of functions $f_n(x)$ is defined by

$$f_1(x)=(\tfrac{1}{4}+\tfrac{1}{2}\mu)f(x)-\tfrac{1}{2}xf'(x),$$
$$f_n(x)=(\tfrac{1}{4}+\tfrac{1}{2}\mu+n-1)f_{n-1}(x)-\tfrac{1}{2}xf'_{n-1}(x),$$

then the author proves that $f_n(x)$ is $R_{\mu+2n}$. The paper includes additional properties of functions of the type $f(x)$, but it may be noted that the author's proof that $-xd[xf(x)]/dx$ is R_μ is not valid.

M. C. Gray.

Polynomials, Polynomial Approximations

Weisner, Louis. Roots of certain classes of polynomials. Bull. Amer. Math. Soc. 48, 283–286 (1942). [MF 6407]

The author starts from Hermite's theorem: If the roots of $\phi(z)=0$, $f(z)=0$ ($\phi(z)$, $f(z)$ polynomials) are all real, all roots of $\phi(D)f(z)$ are also real ($D=d/dz$). The following results are obtained. (I) If all roots of $f(z)=\sum c_k z^k=0$ satisfy $|z|=1$ and all roots of $\phi(z)=0$ satisfy $\Re(z)=0$, then

all roots of

$$g(z)=\sum c_k\cdot\phi(2k-n)z^k=0$$

(n=degree of $f(z)$) satisfy $|z|=1$. (II) If all roots of $\phi(z)=0$ satisfy $\Re(z)=0$ and all roots of $f(z)=0$ satisfy $r_1\leqq|z|\leqq r_2$, then all roots of $g(z)=0$ also satisfy $r_1\leqq|z|\leqq r_2$. Two corollaries to (II) are obtained by specializing (1) $r_1=r_2$; (2) $f(z)=1+z+\cdots+z^n$. (III) If all roots of $\phi(z)=0$ satisfy $\Re(z)\leqq 0$ and all roots of $f(z)=0$ satisfy $|z|\leqq r$, then all roots of $g(z)=0$ satisfy $|z|\leqq r$. For the main result (I) two proofs are presented, one algebraic, the other involving the exponential function. *A. J. Kempner* (Boulder, Colo.).

Lahaye, Edm. Sur la représentation des racines des équations algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 418–427 (1941). [MF 6894]

Let $f(x)=x^p+a_1x^{p-1}+\cdots+a_p$ be a polynomial with only simple zeros and let $F(x,t)=x^p+a_1tx^{p-1}+\cdots+a_pt^p+kt^p-k$. The zeros x_q of $F(x,t)$ may be represented about $t=0$ by the series

$$x_q=\exp(2\pi qi/p)k^{1/p}-a_1t/p+\cdots,\quad q=1,2,\cdots,p,$$

with $k^{1/p}$ any one pth root of k. The author shows it possible to choose the parameter k as an algebraic function of the coefficients $a_1,a_2,\cdots,a_p$ only so that no critical point of equation $F(x,t)=0$ will lie in a strip enclosing the positive axis of reals in the t-plane and hence that the above series may be extended to $t=1$. Since $F(x,1)=f(x)$, series for the zeros of $f(x)$ are thus obtained. *M. Marden.*

Lipka, Stephan. Über die Abzählung der reellen Wurzeln von algebraischen Gleichungen. Math. Z. 47, 343–351 (1941). [MF 6721]

If a real polynomial $F(x)$ of degree n has V variations of sign in its coefficients, p positive zeros and P zeros in the sector $|\arg x|\leqq\pi/n$, then by Descartes' rule $p\leqq V$ and by Obreschkoff's rule [Jber. Deutsch. Math. Verein. 33, 52–64 (1924)] $P\leqq V$. In the present paper Lipka derives sufficient conditions for the equality sign in these two rules, by using two lemmas based on Schoenberg's theorem on variations-diminishing linear transformations [Math. Z. 32, 321–328 (1930)]. Lipka proves that $p=V$ if the nonreal zeros of $F(x)$ lie in the sector $|\arg(-x)|\leqq\pi/n$ or in the sector $|\arg(-x)|\leqq\pi/(p+2)$ or if $a\pm bi$ are the only nonreal zeros and $0<-a<2(r_1+r_2+\cdots+r_p)$, where $r_1,r_2,\cdots,r_p$ are the positive zeros. He shows, furthermore, that $P=V$ if all the zeros of $F(x)$ lie in the double sector $|\arg\pm x|\leqq\pi/n$. *M. Marden* (Milwaukee, Wis.).

Perlin, Irwin E. Sufficient conditions that polynomials in several variables be positive. Bull. Amer. Math. Soc. 48, 458–466 (1942). [MF 6714]

Let $L(x,y)=\sum c_{ij}x^iy^j$ be a polynomial of degree $2m$ in x and of degree $2n$ in y, and let c_{ij} be real. Sufficient conditions are obtained in order that the polynomial be positive for all real values of the variables. These conditions are stated as relations between the coefficients. Similar results are obtained for polynomials in several variables.

A. C. Schaeffer (Stanford University, Calif.).

Erdös, P. and Szegö, G. On a problem of I. Schur. Ann. of Math. (2) 43, 451–470 (1942). [MF 7002]

Let $Q_n(x_0)$ be the class of polynomials $f(x)$ of degree n which are bounded by 1 in the interval $(-1,1)$ and satisfy $f''(x_0)=0$. Let m_n be the maximum of $n^{-2}|f'(x_0)|$ when x_0 ranges over $(-1,1)$. It is shown that this maximum of $n^{-2}|f'(x_0)|$ is attained if and only if $x_0=\pm1$ and $f(x)$ satisfies the differential equation of Zolotareff. This differential equation contains three parameters, which, when the solution is a polynomial, are implicitly related by two elliptic integrals. It is further shown that the limit of m_n as n becomes infinite exists, and the value $(0.3124\cdots)$ of this limit is obtained in terms of elliptic integrals. These results are a continuation of the work of I. Schur, who obtained approximate bounds for m_n by more elementary methods.

A. C. Schaeffer (Stanford University, Calif.).

Jackson, Dunham. Generalization of a theorem of Korous on the bounds of orthonormal polynomials. Bull. Amer. Math. Soc. 48, 602–608 (1942). [MF 7056]

Extending a theorem of Korous [see, for example, Szegö, Orthogonal Polynomials, American Mathematical Society, New York, 1939, p. 157], the author shows that, if $\{p_{n\sigma}(x,y)\}$ is a uniformly bounded set of polynomials orthonormal with weight function $\rho(x,y)$ on an algebraic curve (belonging to a specified wide class of curves), then the set $\{q_{n\sigma}(x,y)\}$, orthonormal on the same curve with weight function $\rho(x,y)\sigma(x,y)$, is also uniformly bounded if $\sigma(x,y)$ is bounded from zero and satisfies a Lipschitz condition on the curve.

E. S. Pondiczery (Princeton, N. J.).

Feldheim, E. Formules d'inversion et autres relations pour les polynomes orthogonaux classiques. Bull. Soc. Math. France 68, 199–228 (1940). [MF 6815]

If a sequence of polynomials $P_n(x)=\sum_{k=0}^n a_{n,k}x^k$ is given, the corresponding inversion formula is $x^n=\sum_{k=0}^n b_{n,k}P_k(x)$. The author gives a compilation of such formulas for the classical orthogonal polynomials as well as Appell and Bernoulli polynomials. Many of these relations are scattered in the literature and the author could have enhanced the value of his contribution by attempting to supply bibliographical references. He gives a number of different methods for the derivation of inversion formulas some of which are symbolical. He is particularly successful in deriving formulas of the type $\lambda^nP_n(x/\lambda)=\sum_{k=0}^n c_{n,k}(\lambda)P_k(x)$, which give the inversion formula when $\lambda\to0$. *E. Hille.*

Rabini, José. On some properties of the derivatives and on certain primitives of Legendre polynomials. Union Mat. Argentina, Publ. no. 20, 8 pp. (1941). (Spanish) [MF 7135]

Identical with the paper in Revista Union Mat. Argentina 7, 65–70 (1941) [see these Rev. 3, 112].

Shohat, J. Note on closure for orthogonal polynomials. Bull. Amer. Math. Soc. 48, 488–490 (1942). [MF 6718]

Let $p(x)$ be a function defined in $(-h,+h)$. Denote by $\phi_n(x)$ the sequence of polynomials orthogonal with respect to $p(x)$. Assume that $p(x)$ is such that the Parseval formula holds, that is,

$$\int_{-h}^{+h}f^2(x)p(x)dx=\sum_{n=0}^{\infty}f_n^2,\quad f_n=\int_{-h}^{+h}f(x)\phi_n(x)p(x)dx.$$

Put $p_1(x)=x^{-1}p(x^{\frac12})$, $p_2(x)=xp_1(x)$. These functions are defined in $(0,h^2)$. The author proves that the Parseval formula also holds for the polynomials orthogonal with respect to $p_1(x)$ and $p_2(x)$. The author applies this theorem for the Hermite and Laguerre polynomials. *P. Erdös.*

Dimsdale, Bernard. Approximation of continuous functions by means of lacunary polynomials. Bull. Amer. Math. Soc. 48, 608–617 (1942). [MF 7057]

D. Jackson's theorems on the degree of approximation to continuous functions by polynomials [Amer. Math. Soc.

Colloquium Publ., vol. 11, New York, 1930, pp. 13–18] are generalized to the case of approximation by functions of the form $P_{\mu_n}(x)=\sum_{i=0}^{n}a_{\mu_i}x^{\mu_i}$, where the μ_i's are non-negative but not necessarily integral and tend monotonically to infinity, and the gaps $\mu_{i+1}-\mu_i$ are bounded. For integral μ_i sufficient conditions are also given for uniformly convergent expansions in series of functions of the form $P_{\mu_n}(x)$ obtained by orthogonalizing the set of powers $\{x^{\mu_i}\}$ with respect to a given weight function.

P. W. Ketchum (Urbana, Ill.).

Differential Equations

Butlewski, Z. Sur les zéros des intégrales réelles des équations différentielles linéaires. Mathematica, Timişoara 17, 85–110 (1941). [6733]

L'auteur démontre pour le système différentiel du premier ordre

$$dx/dt+a(t)x+b(t)y=0,\quad dy/dt+c(t)x+d(t)y=0$$

des théorèmes essentiellement équivalents aux théorèmes de Sturm sur les zéros des solutions d'une équation linéaire du second ordre, le fait caractéristique étant que l'on évite les hypothèses de dérivabilité des coefficients qui seraient nécessaires pour passer du système à l'équation du second ordre par l'étude préliminaire d'une fonction auxiliaire, formée par combinaison linéaire des fonctions inconnues x, y. La note est une traduction et généralisation du premier chapitre d'un travail précédent [Wiadomosci Matematyczne 54, 17–81 (1938) (Polish)]. Dans les §§ 2, 3 on démontre analoguement quelques théorèmes de comparaison entre deux systèmes du type considéré. *B. Levi* (Rosario).

Segond, Marcel. Sur l'intervalle de convergence dans la méthode de Cauchy-Lipschitz. J. Math. Pures Appl. (9) 20, 339–346 (1941). [MF 6838]

This paper is concerned with some elementary theorems on the extensibility of solutions of a first order differential equation $y'=f(x, y)$, and a corresponding system of two equations $y'=f(x, y, z)$, $z'=g(x, y, z)$, where $f(x, y)$ and $f(x, y, z)$, $g(x, y, z)$ are merely assumed to be continuous.

W. T. Reid (Chicago, Ill.).

Bögel, K. Das Verhalten gedämpfter und aufschaukelnder freier Schwinger unter der gleichzeitigen Einwirkung einer konstanten Reibungskraft. Ing.-Arch. 12, 247–254 (1941). [MF 6425]

The author considers the equation of motion of an oscillator subject to viscous and Coulomb damping:

$$a\ddot{x}+b\dot{x}+cx+(\operatorname{sgn}\dot{x})R=0,$$

where a, b, c, R are constants, including the negative damping case $b<0$. The equation is solved over time intervals between reversals of sign of $\dot{x}$, and the constants of one solution are related to those of the following one. A geometrical construction is outlined which enables one to tell at a glance what the limiting nature of the motion and of the maximum displacement is. *H. Poritsky.*

Levinson, Norman and Smith, Oliver K. A general equation for relaxation oscillations. Duke Math. J. 9, 382–403 (1942). [MF 6876]

The authors deal with the nonlinear differential equation

$$(*)\qquad \ddot{x}+f(x, \dot{x})\cdot\dot{x}+g(x)=0,\quad \dot{x}=dx/dt=v.$$

They derive conditions on $f(x, v)$ and $g(x)$ under which a periodic solution of (*) exists and other conditions which ensure that a unique periodic solution exists. In the latter case it is then readily seen that all solutions tend, as $t\to\infty$, to the unique periodic solution. The equation (*) is a generalization of the differential equation:

$$(**)\qquad \ddot{x}+\epsilon(x^2-1)\cdot\dot{x}+x=0,\qquad \epsilon>0,$$

which was first derived and studied by B. van der Pol in connection with electrical problems involving triode vacuum tubes. The question of existence of periodic solutions in a case somewhat more general than that of (**) has been considered by E. and H. Cartan [Ann. Post. Télégr. Téléph. 14 (1925)]. The more difficult uniqueness problem has been solved by Liénard [Rev. Gén. de l'Elect. 23, 901–946 (1928)] in a case much less general than that of (*). (It might be noted that the term relaxation oscillation is generally applied in practice to certain special limit cases, of which the case $\epsilon\to\infty$ in (**) is typical.)

The essential restrictions on $f(x, v)$ and $g(x)$ for the proof of the existence of a periodic solution of (*) are simply that $xg(x)\geqq 0$ and that $f(x, v)$ is positive for large $|x|$ and negative for small $|x|$ and $|v|$. The differential equation is reduced to one of first order and the existence proof is carried out in the x, v-plane. It involves the application of a well-known theorem of Bendixson to prove that a closed solution curve, or cycle, exists, the main difficulty being to show that a bounded solution occurs. The possibility of proving by such means the existence of a periodic solution in cases like the present one where only one singularity is involved has been pointed out by G. D. Birkhoff [Dynamical Systems, American Mathematical Society Colloquium Publications, 1927, New York, chap. 5]. The proof of uniqueness of the cycle is made to depend upon considerations of stability. The authors prove that every cycle is stable in the sense that every integral curve starting near enough to the cycle tends to it as $t\to+\infty$. Hence every cycle is isolated and, since two successive ones could not both be stable, it follows that at most one could exist. The authors establish the stability of a cycle by methods which are of interest for their own sake. Essential use is made of the restrictions on the functions f and g. *J. J. Stoker.*

Caccioppoli, Renato. Esistenza e limitazione dello spettro in un problema ai limiti per un'equazione differenziale ordinaria non lineare. Portugaliae Math. 3, 79–86 (1942). [MF 6770]

This paper is concerned with the determination of bounds on the values of the parameter λ for which the nonhomogeneous second order differential equation

$$\omega(y, y', y'')y''/(1+y'^2)^{3/2}+\lambda y=0$$

has a solution $y=y(x)$, $-1\leqq x\leqq+1$, which is positive on the interior of this interval and vanishes at the end-points. The analysis is based upon the following hypotheses: for (y, y', y'') ranging independently on $(-\infty, +\infty)$, the function $\omega(y, y', y'')$ is continuous, is an even function of y' for arbitrary values of (y, y'') and has its least upper and greatest lower bounds finite and positive. *W. T. Reid.*

Wittich, Hans. Ganze Lösungen der Differentialgleichung $w''=f(w)$. Math. Z. 47, 422–426 (1941). [MF 6727]

If $f(w)$ is an integral transcendental function and $p_i(z)$ are polynomials, the paper shows that the differential equation

$$(*)\qquad w^{(n)}(z)+p_1(z)w^{(n-1)}+\cdots+p_{n-1}(z)w'(z)=f(w),\qquad n\geqq 2,$$

can not have an integral function $w=w(z)$ for a solution. If $f(w)$ is a polynomial of degree $q\geqq n+1$, then any integral function satisfying (*) is a polynomial. If $f(w)$ is an integral but not linear function and all $p_i(z)\equiv 0$, then (*) has no solution in the class of integral functions [F. Rellich, Math. Z. 47, 153–160 (1940); these Rev. 3, 241, proved this last theorem for the equation $w''=f(w)$]. In the above theorems the constant solutions $w=c$, where $f(c)=0$, are implicitly excluded. *F. G. Dressel* (Durham, N. C.).

Ascoli, Guido. Sulla forma asintotica degli integrali dell'equazione differenziale $y''+A(x)y=0$ in un caso notevole di stabilità. Univ. Nac. Tucumán. Revista A. 2, 131–140 (1941). [MF 6753]

For the differential equation

$$y''+[A(x)+\eta(x)]y=0 \tag{I}$$

the author obtains asymptotic forms for an arbitrary solution y, and its derivative y', under the following hypotheses: (i) there is an interval $c\leqq x<+\infty$ on which the real-valued function $A(x)$ is of bounded variation; (ii) the function $\eta(x)$ is absolutely integrable on this interval; (iii) $A(x)$ tends to a positive limit as x approaches $+\infty$. In particular, under these hypotheses it follows that y and y' are bounded on $c\leqq x<+\infty$. In addition, it is proved that, if $A(x)$ is such that all solutions y of $y''+A(x)y=0$ are bounded on an interval $c\leqq x<+\infty$, and $\eta(x)$ is absolutely integrable on this interval, then all solutions y of the above equation (I) are also bounded on this interval. *W. T. Reid.*

Jeffreys, Harold. Asymptotic solutions of linear differential equations. Philos. Mag. (7) 33, 451–456 (1942). [MF 7070]

The equation (†) $d^2y/dx^2=(h^2X_0+hX_1+X_2)y$, where h is independent of x and the X_i's are given functions of x, with $X_0(0)=0$, is approximated for large h and small x by an equation of the type (*) $d^2y/dz^2=zy$. Two solutions of (*) are

$$y_1=\pi^{-1}\int_0^\infty \cos\left(\tfrac{1}{3}t^3+zt\right)dt$$

(the "Airy integral") and

$$y_2=\pi^{-1}\int_0^\infty \left[\exp\left(-\tfrac{1}{3}t^3+zt\right)+\sin\left(\tfrac{1}{3}t^3+zt\right)\right]dt.$$

The author uses y_1 and y_2 to derive asymptotic expressions of the solutions of (†) which are of exponential type on one side of $x=0$ and oscillatory on the other. Such solutions occur when $x=0$ is a simple zero of X_0. If the argument is not large enough for the asymptotic expansions to be valid the use of the tables of the Airy integral, which are being published by the British Association, is suggested.

I. Opatowski (Chicago, Ill.).

Pougatcheff, V. Sur les représentations asymptotiques des intégrales des systèmes d'équations linéaires contenant un paramètre. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 431–439 (1941). (Russian. French summary) [MF 6833]

Let $\{x, \alpha\}$ denote generically a formal series

$$a_0(x)+a_1(x)\alpha^{-1}+\cdots,$$

where the coefficients are functions of x differentiable a suitable number of times on (a, b) $(a\leqq x\leqq b)$; α is a parameter in a domain D extending to infinity. Let $[x, \alpha]^\mu$ be generically a function $\sim$ (asymptotic, uniformly with respect to x, to) $\{x, \alpha\}$ to μ terms. The author studies a matrix differential equation (1) $y^{(1)}=yp+\pi$, where (2) $p=\alpha^r[x, \alpha]^{m+r}$ $(m\geqq 0)$, (3) $\pi=\alpha^s z[x, \alpha]^{m+r-l}$ $(0\leqq l\leqq r)$, while z is a solution of (4) $z^{(1)}=zq$ $(q=\alpha^r[x, \alpha]^{m+r})$. It is assumed that the characteristic numbers of the matrix $\alpha^{-r}p$ $(\alpha=\infty)$ are distinct on (a, b). By a certain substitution (5) $y=u\sum_0^r s_\nu(x)\alpha^{-\nu}$ equation (6) $y^{(1)}=yp$ goes into (7) $u^{(1)}=u(\lambda+h)$ $(h\sim 0)$, where λ is a diagonal matrix with elements $\lambda_i^*=\sum_0^r \lambda_i^{(\nu)}(x)\alpha^{r-\nu}$. The $\lambda_i=\sum_0^{r-1}$ . . . are termed c.p.'s (characteristic polynomials) of (6). The coefficients in the c.p.'s of (4) are designated by $\mu_i^{(\nu)}(x)$. System (6) is said to be of class A when $\Re(\lambda_i-\lambda_j)$ has a finite upper or lower bound. Under certain, precisely stated, differentiability conditions on the coefficients in (1), (4) and under the assumption that (6), (4) are of class A the following is proved. (A) If the c.p.'s of (6) and (4) are the same then (6) has a solution $y=z[x, \alpha]^m$. (B) The system (1) has a solution $y=\alpha^{s+l-r}z[x, \alpha]^m$ $(l=\max l_{i,j})$ if $\mu_i^{(\nu)}-\lambda_j^{(\nu)}\equiv 0$ $(\nu\leqq l_{i,j}-1)$, while $\mu_i^{(l_{i,j})}-\lambda_j^{(l_{i,j})}$ has no zeros on (a, b) $(i, j=1, \cdots, n)$. *W. J. Trjitzinsky* (Urbana, Ill.).

Mammana, Gabriele. Autofunzioni relative a sistemi differenziali contenenti una condizione quadratica in due punti. Nota preventiva. Anais Acad. Brasil. Sci. 12, 35–37 (1940). [MF 6569]

The results herein abstracted extend and make more precise some of the theorems previously established by the author [Ann. Mat. Pura Appl. (4) 19, 81–106 (1940); these Rev. 2, 226] for a special second-order differential system involving boundary conditions which are quadratic inequalities in the end-values at two points of the solution function and its derivative. *W. T. Reid* (Chicago, Ill.).

Mękai, E. Asymptotische Abschätzung der Eigenwerte gewisser Differentialgleichungen zweiter Ordnung. Ann. Scuola Norm. Super. Pisa (2) 10, 123–126 (1941). [MF 6418]

By considerations based upon a change of variable, it is shown very simply that the characteristic values λ_n of the equation $y''+\lambda\rho(x)y=0$ ($\rho(x)$ positive and of class C''), together with either (a) $y(a)=y(b)=0$, or (b) $y'(a)=y'(b)=0$, are given asymptotically by the formula

$$\lambda_n^{\frac{1}{2}}-n\pi/\int_a^b(\rho(x))^{\frac{1}{2}}dx=O(1/n),$$

and that those of the equation $y''+[\lambda+p(x)]y=0$, together with either (a) or (b), are such that

$$\int_a^b(\lambda_n+p(x))^{\frac{1}{2}}dx-n\pi=O(1/n^2).$$

R. E. Langer (Madison, Wis.).

Ince, E. L. Simultaneous linear partial differential equations of the second order. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 195–209 (1942). [MF 7095]

This paper, which was edited by Erdélyi after Ince's death, deals with a pair of partial differential equations

$$ar+bc_1s+dp+e_1q+fz=0,$$
$$a_1t+b_1cs+d_1q+cp+f_1z=0,$$

in which the coefficients without suffix are functions of x only, those with suffix functions of y only. These functions are supposed, moreover, for simplicity to be polynomials chosen in such a way that the two equations are compatible and so have at least one analytic solution z which does not vanish identically. The equations under consideration be-

long to the system in which the coefficients are polynomials in the two variables, a system which seems to have been discussed only by Horn. In this study a general condition for consistency is worked out and some particular cases considered. The system found when certain restrictions are introduced is transformable into a hypergeometric system. *H. Bateman* (Pasadena, Calif.).

Lahaye, Edm. Sur l'application de la méthode des approximations successives à la résolution des équations aux dérivées partielles linéaires du second ordre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 537–551 (1941). [MF 6905]

Let the second order partial differential equation

$$(1)\qquad \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}+c(x)u=0,$$

$$f(x)\equiv f(x_1, \cdots, x_n),$$

have solutions of the form

$$(2)\quad u=R_1(x)F_1(\varphi_1(x), \cdots, \varphi_{n-1}(x)) +R_2(x)F_2(\psi_1(x), \cdots, \psi_{n-1}(x)),$$

where R_1, R_2, φ_i, ψ_i are known functions and F_1 and F_2 are arbitrary functions. When the initial conditions for (1) are Cauchy's, or similar ones, the paper presents a method of determining the F_1 and F_2 so as to satisfy these boundary conditions. If equation (1) does not have the type of solutions indicated in (2), the author suggests a method of successive approximations for handling the boundary value problems. The details of the method of successive approximations are promised in another paper. *F. G. Dressel.*

Robinson, Lewis-Bayard. Un système de Riquier et le calcul tensoriel. II. Bull. Soc. Math. France 68, 129–133 (1940). [MF 6811]

[For the first part, cf. C. R. Acad. Sci. URSS. (N.S.) 15, 411–414 (1937).] For a system of two linear homogeneous differential equations of the second order, Wilczynski [Projective Differential Geometry of Curves and Ruled Surfaces, Teubner, Leipzig, 1906, chapter 4] has given its semi-covariants. The proofs are based on the Lie theory of infinitesimal transformations. Using similar methods, the present paper gives the form of the semi-tensors of the same system of differential equations. *F. G. Dressel.*

Ostrowski, Alexandre. Sur une classe de transformations différentielles dans l'espace à trois dimensions. II. Comment. Math. Helv. 14, 23–60 (1942).

[Part I appeared in the same Comment. 13, 156–194 (1941); these Rev. 3, 43.] If $y_i(x)$, $i=1, 2$, are two indeterminant functions of a single variable x, transformations (1) $\xi=\xi(x, y_i, r)$, $\eta_j=\eta_j(x, y_i, r)$, $r=r(x, y_i, dy_i/dx)$, $j=1, 2$, such that $x=x(\xi, \eta_i, \rho)$, $y_j=y_j(\xi, \eta_i, \rho)$, $\rho=\rho(\xi, \eta_i, d\eta_i/d\xi)$, are called by the author R transformations. If the function r is eliminated from (1), two relations of the form (2) $\Omega_j(x, y_i; \xi, \eta_i)=0$ are obtained. The present paper is devoted to the problem of finding R transformations when one starts with a couple of relations of the type (2). The principal results obtained are that all R transformations, except for certain singular ones, can be obtained from a couple of relations (2). The singular R transformations are easily characterized and can be obtained from a single relation of the type (2). *F. G. Dressel* (Durham, N. C.).

Soudan, Robert. Sur la déformabilité d'un corps à potentiel constant. C. R. Soc. Phys. Genève 57, 79–82 (1940). [MF 6488]

Consider the logarithmic potential due to a homogeneous simple distribution of mass M on a plane lamina S. Given any positive constant ϵ and any positive integer n, there exist a constant A and an infinitesimal transformation displacing each point $P:(\rho, \theta)$ of the boundary of S by an amount $i(\theta)\epsilon$ such that, if the mass M is distributed uniformly on the transformed lamina, then at all points at a sufficient distance from S the variation in the potential is at most $\epsilon^n A$. *F. W. Perkins* (Hanover, N. H.).

Wavre, Rolin. À propos d'un problème d'attraction et les fonctions orthogonales aux fonctions harmoniques. C. R. Soc. Phys. Genève 57, 113–115 (1940). [MF 6489]

The author studies the functions which are orthogonal to all harmonic functions in a domain D and proves their existence. He also proves the decomposition theorem: If u is continuous and $\int u^2 dD$ is finite, then $u=H+f$, where H is harmonic and f is orthogonal to all harmonic functions in D with integrable squares; also this decomposition is unique. The method used is to minimize $\int(u-h)^2 dD$, where h ranges over harmonic functions with bounded $\int h^2 dD$. If the minimizing function h is H, and $u-H=f$, then f has the desired orthogonal property. *J. W. Green.*

Sadosky, Manuel. On the boundary behavior of the derivative of a harmonic function. Revista Union Mat. Argentina 8, 70 (1942). (Spanish) [MF 7255]

An example is given to show that, if $u(x, y)$ is harmonic for $x^2+y^2<1$ and continuous for $x^2+y^2\leqq 1$, if du/ds exists and is bounded on $x^2+y^2=1$, where s is the boundary arc, then it does not necessarily follow that the directional derivative is bounded in every direction on the interior. *E. F. Beckenbach* (Austin, Tex.).

Weinberg, Alvin M. Green's functions in biological potential problems. Bull. Math. Biophys. 4, 107–115 (1942). [MF 7108]

In the electrophysiology of nerve and muscle the following problem occurs: let S be a closed surface (for example, an electrically polarized membrane covering a tissue) and V another surface surrounding S (for example, the vessel in which the tissue is immersed), to find a function u_e harmonic in the space between S and V, and a function u_i harmonic inside S in such a manner that

$$(1)\qquad u_e-u_i=\Phi_e(\partial u_e/\partial n_e)=\Phi_i(\partial u_i/\partial n_i)$$

on S, n_e and n_i being the normals of S and Φ_e, Φ_i being known functions; (2) $\partial u_e/\partial n_e=0$ or $u_e=0$ on V. The author gives a Green's function suitable for this problem and calculates as an application the steady state potential of a circular cylinder with a uniform surface polarization for $0<z<\infty$ and no polarization for $-\infty<z<0$; Φ_e and Φ_i are then linear functions everywhere except at $z=0$, where a finite jump occurs. This example may be considered to represent a nerve fiber in which an injury current is flowing. *I. Opatowski* (Chicago, Ill.).

Hadamard, J. On the Dirichlet problem for the hyperbolic case. Proc. Nat. Acad. Sci. U. S. A. 28, 258–263 (1942). [MF 6683]

Consider (1) $u_{xy}-(\log \lambda(x, y))_x u_y=0$; $\lambda \not\equiv f(x)g(y)$; (2) Γ a curve symmetric about $x=0$, convex to both axes and with four vertices (a vertex is a maximum or minimum of x or y

on Γ); (3) $u=u|_\Gamma$ assigned on Γ. The well-known general solution of (1) is $u=X(x)+\int\lambda Y(y)dy$. The Dirichlet problem above rests on the determination of Y. The writer indicates that, for fairly general λ, only two conditions on λ and $u|_\Gamma$ at the x vertices are both necessary and sufficient for the solution of (1), (3). These interesting results are somewhat surprising. Thus, in contrast, for (1)′ $u_{xy}=0$, the conditions are known to be of a Diophantine nature on Γ. Moreover, in a paper overlooked by the writer, the reviewer has shown [Duke Math. J. 7, 97–120 (1940); these Rev. 2, 290] that, even for the general constant coefficient hyperbolic equation in two variables with data on a rectangle skew to the characteristics (and for a class of transforms of this formulation), the conditions are again of Diophantine type. *D. G. Bourgin* (Urbana, Ill.).

Hadamard, J. The problem of diffusion of waves. Ann. of Math. (2) **43**, 510–522 (1942). [MF 7005]

Consider

$$(1)\qquad G(u)=A^{ij}\frac{\partial^2 u}{\partial x^i\partial x^j}+B^i\frac{\partial u}{\partial x^i}+Cu,\quad i,j=1,\cdots,4,$$

$A^{ij}=0, 1, -1$ according as $i\neq j$, $i=j\leqq 3$ or $i=j=4$. Write $L(u)$ for the adjoint and consider the group (E) of transformations taking $L(u)$ into $\lambda^{-1}L(\lambda u)$, $\lambda\neq 0$. The equations $\lambda^{-1}L(\lambda u)=0$ constitute an equivalence class. Mathisson [Acta Math. 71, 249–282 (1939); these Rev. 1, 120] showed by use of the Hilbert parametrix that the only class admitting "pure" waves is that containing the representative for which B^i, $C=0$. The present paper reproduces Mathisson's fundamental relations but derives them in a natural way from the elementary solution. Thus, if $\Gamma(x;a)$ is the geodesic distance from the pole $a^1,\cdots,a^4$ to $x^1,\cdots,x^4$, the elementary solution for (2) $G(u)=0$ is (3) $V_0/\Gamma+V_1\log\Gamma$. It is well known that for pure waves the $\log\Gamma$ term cannot appear and this is tantamount to the condition (4) $G(V_0)=0$. The reductions involve the fundamental invariant (under E) curl B^i. At the pole a, the $\{B^i\}$ and their partials undergo transformations (under E) determined by the partials of λ. A convenient canonical form is obtained by choosing $\lambda(x;a)=V_0$. The partials of this λ are expressible in terms of the original $\{B^i\}$ or $\{B_i\}$ and the derivatives of these quantities at a. For instance, $((2/\lambda)\partial\lambda/\partial x^i+B_i)|_a=0$, etc. On introducing (4) and the three (differential) invariants involving C and its first two partial derivatives it develops straightforwardly that curl $B^i=0$. Hence, for suitable λ, the $\{B_i\}$ in $\lambda^{-1}L(\lambda u)$ are zero. From this fact, from the form of the invariant involving C and from (2), it follows that $C=0$. Hadamard shows, also, that if B^i, C are complex functions the conclusions stated above no longer follow.

D. G. Bourgin (Urbana, Ill.).

Bickley, W. G. The effect of a free surface on compressional (shock) waves. Proc. Roy. Soc. London. Ser. A. **180**, 209–218 (1942). [MF 6929]

The author considers a channel of uniform depth h and of uniform breadth, which is extended to infinity in both directions. It is assumed to be filled with a fluid. A partition is located at $x=0$, which divides the channel into two parts $(0<x<\infty)$ and $(-\infty<x<0)$. At the time $t=0$ the partition begins to move in the positive direction with the velocity V. Supposing that the y axis coincides with the vertical direction and introducing the variables $\xi=\pi x/2h$, $\eta=\pi y/2h$, $\tau=\pi ct/2h$, the author obtains for the (nondimensional) velocity potential $\phi=(2hV/\pi)^{-1}\phi'$ the equation $\phi_{\xi\xi}+\phi_{\eta\eta}=\phi_{\tau\tau}$ under certain boundary and initial conditions. He gets

$$\omega=\partial\phi/\partial\tau=8\pi^{-2}\sum m^{-1}\sin m\eta\int_0^{\frac{1}{2}\pi}\cos(m\beta\sin\theta)d\theta,$$

where $\beta=(\tau^2-\xi^2)^{\frac{1}{2}}$. Applying this result the author determines the type of waves which are generated by the motion of the partition, the pressure distribution and the form of the free surface. The method can be generalized to the case where the partition has a nonuniform velocity as well as to the case where the depth h is infinite. *S. Bergman.*

Owens, Glynn. An explicit formula for the solution of the ultrahyperbolic equation in four independent variables. Duke Math. J. **9**, 272–282 (1942). [MF 6868]

The author considers the equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}-\frac{\partial^2 u}{\partial x_3^2}-\frac{\partial^2 u}{\partial x_4^2}=0.$$

Let u be a sufficiently regular solution in the region bounded by the characteristic cone with vertex P and a surface C. Then $u(P)$ is expressed explicitly in terms of the values of u and of a finite number of its derivatives on C. That expression is derived by the use of a generalization of Riemann's method due to H. Lewy [Nachr. Ges. Wiss. Göttingen. Math. Phys. Kl. **1928**, 118–123], in which a system of Riemann functions takes the place of the single one of the two-dimensional case. Necessary and sufficient conditions for the existence of a solution u to given initial values on C are not known. The author derives two necessary conditions in the form of integro-differential equations, and shows by an example that those conditions are not identically satisfied. *F. John* (Lexington, Ky.).

Tranter, C. J. The application of the Laplace transformation to a problem on elastic vibrations. Philos. Mag. (7) **33**, 614–622 (1942). [MF 7225]

Let $U(r,t)$ denote the radial displacement at time t of a point at a distance r from the axis of a hollow cylinder $r_0<r<r_1$. The length of the cylinder is infinite, and the pressure on the surfaces $r=r_0$ and $r=r_1$ is radial and uniform at each instant. Then U satisfies the differential equation $U_{rr}+U_r/r-U/r^2=U_{tt}/a^2$, where the constant a^2 is a function of the elastic coefficients of the material. The author solves this equation under the conditions that the pressure is zero at $r=r_1$ and $A(1-\cos\omega t)$ at $r=r_0$, with the cylinder initially at equilibrium. His solution has the form

$$U=Ar+B/r^2+F(r)\cos\omega t+\sum_1^\infty F_n(r)\cos\alpha_n t,$$

where $F(r)$ and $F_n(r)$ are linear combinations of Bessel functions and ω is assumed to be distinct from any of the natural frequencies α_n. The corresponding problem for the hollow sphere is also solved, again by direct application of the Laplace transformation. The author points out how the hoop stresses, which are of primary practical importance in such problems, can be obtained from his formulas for displacements. *R. V. Churchill* (Ann Arbor, Mich.).

Green, George. Corresponding problems in periodic and steady flow. Philos. Mag. (7) **33**, 161–173 (1942). [MF 6446]

A continuation of an earlier paper [Philos. Mag. (7) **33**, 102–114 (1942); these Rev. **3**, 246], consisting of further illustrative problems. *D. G. Bourgin* (Urbana, Ill.).

Jaeger, J. C. Radial heat flow in circular cylinders with a general boundary condition. II. J. Proc. Roy. Soc. New South Wales 75, 130–139 (1942). [MF 6680]

[The first part appeared in the same J. Proc. 74, 342–352 (1940); cf. these Rev. 2, 204.] By a formal use of the Laplace transformation in the first part of this paper the author obtained solutions of several temperature problems for regions bounded internally or externally by cylinders with a general boundary condition at the surface. In part II the author first derives essential properties of the zeros of certain combinations of Bessel functions. He is then able to show rigorously that the integrals and series used to represent his temperature functions converge and that these functions satisfy all the conditions in the boundary value problems. *R. V. Churchill* (Ann Arbor, Mich.).

Jaeger, J. C. Heat conduction in a wedge, or an infinite cylinder whose cross-section is a circle or a sector of a circle. Philos. Mag. (7) 33, 527–536 (1942). [MF 7089]

Using Laplace transforms the author derives formulas for the temperatures in the regions $0<\theta<\theta_0$, $r>0$, and $0<\theta<\theta_0$, $0<r<a$ $(0<\theta_0\leqq 2\pi)$, under the conditions of zero initial and constant surface temperatures. Tables and graphs of the temperature functions for some of the most common values of the angle of opening θ_0 are shown. For the cylindrical sectors the points which heat most slowly are found and their temperature variations are shown graphically. It is also pointed out that when the conductivities in the directions $\theta=$constant and $r=$constant are different the problem is reducible to the isotropic cases already treated. *R. V. Churchill* (Ann Arbor, Mich.).

Mersman, W. A. Heat conduction in a semi-infinite slab. Philos. Mag. (7) 33, 303–309 (1942). [MF 6593]

The author treats the boundary-value problem $u_t=u_{xx}$ for $t>0$, $x>0$ with $u(x, 0)=f(x)$ and $u_x(0, t)=g(t)$. Using Laplace transforms a formal solution is obtained which is then rigorously established under general conditions. Finally, the boundary-values $u(0, t)$ are obtained. *W. Feller* (Providence, R. I.).

Pugh, H. Ll. D. and Harris, A. J. The temperature distribution around a spherical hole in an infinite conducting medium. Philos. Mag. (7) 33, 661–666 (1942). [MF 7228]

The problem treated here by Heaviside's operators is essentially the same as one already solved in the literature to illustrate the advantages of using the Laplace transformation. The well-known substitution of a new temperature function $u(r, t)=r\theta(r, t)$ reduces the problem treated here to one of determining the temperature u in a semi-infinite slab with radiating face. The Green's function for the latter problem is given by Carslaw and Jaeger [Philos. Mag. (7) 26, 473–495 (1938), in particular, p. 478]. *R. V. Churchill* (Ann Arbor, Mich.).

Calculus of Variations

***Brady, Carroll Parker. The minimum of a function of integrals in the calculus of variations.** Contributions to the Calculus of Variations, 1938–1941, pp. 1–52. University of Chicago Press, Chicago, Ill., 1942. $3.00.

As a generalization of certain special problems initiated by Euler, the author considers the following problem in the calculus of variations. Consider, in the x, y-plane, a class M of (properly defined) admissible arcs of the form $y=y(x)$, $x_1\leqq x\leqq x_2$. Given n integrands $h_\beta(x, y, y')$, $\beta=1, 2, \cdots, n$, consider the integrals $I_\beta=\int_{x_1}^{x_2}h_\beta(x, y, y')dx$. The problem then consists of minimizing a given function $g[I_1, I_2, \cdots, I_n]$ of these integrals in the subclass of admissible arcs that join two given points in the x, y-plane. This problem may be treated as a special case of the problem of Bolza. Let M' denote the class C of curves in $(n+2)$-space which (in addition to certain assumptions concerning regularity) satisfy the following conditions. (a) C can be represented in the form $y_i=y_i(x)$, $x_1\leqq x\leqq x_2$; $i=0, 1, 2, \cdots, n$. (b) $x_s-a_s=y_0(x_s)-y_{0s}=y_\beta(x_1)=0$, $s=1, 2$; $\beta=1, 2, \cdots, n$. (c) $h_\beta(x, y_0, y_0')-y_\beta'=0$, $\beta=1, 2, \cdots, n$. In this class M', it is required to minimize the function $g[y_1(x_2), \cdots, y_n(x_2)]$. For this problem a comprehensive set of results is derived on the basis of the theory of the general problem of Bolza. *T. Radó* (Columbus, Ohio).

***Nordhaus, Edward Alfred. The problem of Bolza for double integrals in the calculus of variations.** Contributions to the Calculus of Variations, 1938–1941, pp. 53–97. University of Chicago Press, Chicago, Ill., 1942. $3.00.

The writer derives necessary conditions for the problem of Bolza for double integrals. More specifically, let S be a fixed regular surface of class C''' defined by $x=X(u, v)$, $y=Y(u, v)$, $z=Z(u, v)$, defined for $-\infty<u<+\infty$, $-\epsilon\leqq v\leqq\epsilon$, where $X(u, v)$, $Y(u, v)$, $Z(u, v)$ are periodic in u, and suppose S is representable in the form $z=\varphi(x, y)$ (φ of class C''') near the regular projection C_0 of the regular simple closed curve $L_0: x=X(u, 0)$, etc. Let $z=z_0(x, y)$ be of class C'', bounded by L_0, and suppose $z_0(x, y)$ minimizes the functional

$$I(z)=\iint_A f(x, y, z, p, q)dxdy+\int_L g(x, y, z, x', y', z')ds$$

among all surfaces $z=z(x, y)$ of class D' which are bounded by curves L on S which are sufficiently near L_0 and have 1-1 "piecewise regular" projections C on the (x, y) plane, A being the region bounded by C; f and g are of class C''' and g is positively homogeneous in (x', y', z'). The writer discusses the first and second variations of $I(z)$ and derives a transversality condition which must be satisfied by z_0 on C_0 and further boundary inequalities which involve the function g and are analogues of the Weierstrass and Legendre conditions. An application is made to a sort of Plateau problem in which $f=(1+p^2+q^2)^{\frac{1}{2}}$, $g=(x'^2+y'^2+z'^2)^{\frac{1}{2}}$. *C. B. Morrey, Jr.* (Berkeley, Calif.).

***Stokes, Ellen Clayton. Applications of the covariant derivative of Cartan in the calculus of variations.** Contributions to the Calculus of Variations, 1938–1941, pp. 139–174. University of Chicago Press, Chicago, Ill., 1942. $3.00.

In this paper the criteria for a minimizing arc in the parametric problem of the calculus of variations in n-space with fixed end points are expressed in a tensor form involving the covariant derivative of Cartan. *S. B. Myers* (Ann Arbor, Mich.).

***Landers, Aubrey Wilfred, Jr. Invariant multiple integrals in the calculus of variations.** Contributions to the Calculus of Variations, 1938–1941, pp. 175–207. University of Chicago Press, Chicago, Ill., 1942. $3.00.

This paper deals with a generalization of the Hilbert invariant integral to multiple integral problems in the cal-

culus of variations. Necessary and sufficient conditions are given that an integral whose integrand is a function of m independent variables (t), n dependent variables (x) and the derivatives (x_t) be invariant, that is, have the same value for all m-spaces $x(t)$ having the same fixed $(m-1)$-space as boundary. It is proved that when $m>1$ every family of extremal manifolds which simply covers a region F of (t, x) space has an associated invariant integral, which is unique only if $n=1$. A study is made of such fields, and of certain partial differential equations which arise.

S. B. Myers (Ann Arbor, Mich.).

*Landers, Mary Kenny. **The Hamilton-Jacobi theory for the problems of Bolza and Mayer.** Contributions to the Calculus of Variations, 1938–1941, pp. 209–291. University of Chicago Press, Chicago, Ill., 1942. $3.00.

The author reviews the theory already developed by others for the problem of Bolza in the nonparametric case, and she translates these results to the parametric problem. She defines Mayer fields and introduces the Hamilton-Jacobi equation. Solutions of the Hamilton-Jacobi equation are discussed, and the author gives a method of constructing a family of extremals from a complete integral of the equation. At all times the author keeps the problem of Mayer in mind and calls attention to any noteworthy properties of her analysis in connection with this problem; she compares her results with some obtained by Kneser for the problem of Mayer. The problem of Bolza is a variable end point calculus of variations problem in which admissible arcs are restricted by a set of first order differential equations and in which the sum of a function of the end points and the usual calculus of variations integral along admissible arcs is minimized. If the integral is not present (leaving only the function of the end points to be minimized) the problem becomes Bliss's version of the problem of Mayer. This paper is not intended for beginners, although it does contain a summary of some of Bliss's lectures on the problem.

Opinions expressed or assertions made in this review are not to be construed as representing the views of the Navy Department or the naval service at large.

C. B. Tompkins (Washington, D. C.).

*Hazard, Katharine Elizabeth. **Index theorems for the problem of Bolza in the calculus of variations.** Contributions to the Calculus of Variations, 1938–1941, pp. 293–356. University of Chicago Press, Chicago, Ill., 1942. $3.00.

The writer studies index theorems for a variational problem P determined by a quadratic functional

$$J(\eta)=c_{ij}\eta_{i1}\eta_{j1}+2d_{ij}\eta_{i1}\eta_{j2}+e_{ij}\eta_{i2}\eta_{j2}+\int_{x_1}^{x_2}[P_{ik}(x)\eta_i\eta_k+2Q_{ik}(x)\eta_i\eta_k'+R_{ik}(x)\eta_i'\eta_k']dx$$

of vector functions $(\eta_1, \cdots, \eta_n)$ subject to a set of homogeneous linear differential equations and a set of homogeneous linear equations in the end values $\eta_{i1}=\eta_i(x_1)$ and $\eta_{i2}=\eta_i(x_2)$. This type of problem is suggested by a study of the second variation in the general problems of Bolza and Lagrange. A large part of the paper is devoted to an extension to the problem of Bolza of the theory developed by G. D. Birkhoff and Hestenes [Duke Math. J. 1, 198–286 (1935), in particular, pp. 198–243] in the case where the differential equations are absent.

With each such problem P is associated the linear space E of vector functions of class D' which satisfy all the side conditions. The index $k(P)$ of P is defined as the dimensionality of the maximal linear subspace of E on which $J(\eta)$ is negative definite. It is shown that this index of P (on E) coincides with the number of natural isoperimetric conditions in a minimal set according to the definition used by G. D. Birkhoff and Hestenes. If P^* is a problem determined by the same $J(\eta)$ but E^* is a subspace of E, then P^* is said to be a subproblem of P. The index of P with respect to P^* is defined as $k(P)-k(P^*)$; in special instances, G. D. Birkhoff and Hestenes and Morse have called this the "order of concavity." This idea is used very effectively throughout the paper. The ordinary focal point theory is extended by the introduction of focal intervals and the index is shown to be the number of focal intervals interior to (x_1, x_2) (each being counted a number of times equal to its order). It is also shown that the index defined above is the same as that of Morse in terms of broken extremals and is finite if the "strengthened Clebsch condition" holds. If this strengthened Clebsch condition holds, the index is shown to be the number of negative characteristic values of the characteristic functional $J(\eta)-\sigma\int_{x_1}^{x_2}\eta_i\eta_i dx$. A great many other interesting theorems are proved including a number of oscillation and comparison theorems, one of which is that there are exactly $k-q^0$ conjugate (focal) intervals of x_1 or of x_2 interior to (x_1, x_2), q^0 being the index of P with respect to the fixed end point problem P^0. Finally a "pseudo-periodic" problem is discussed and the relation of the various indices to the "indices of repletion" of Morse and Pitcher [Proc. Nat. Acad. Sci. U. S. A. 21, 282–287 (1934)] is pointed out.

C. B. Morrey, Jr.

*Levin, Joseph Harmon. **Minima of double integrals with respect to unilateral variations and applications to subharmonic functions.** Contributions to the Calculus of Variations, 1938–1941, pp. 357–411. University of Chicago Press, Chicago, Ill., 1942. $3.00.

A harmonic function $h(x, y)$ minimizes the Dirichlet integral $I(u)=\iint(h_x^2+h_y^2)dxdy$ with respect to all (properly defined) admissible functions $v(x, y)$ that satisfy the condition $v(x, y)=h(x, y)$ on the boundary of the region R over which the integration is extended. The Euler-Lagrange equation implied by this minimizing property is the differential equation $\Delta h=0$ of Laplace. Since subharmonic functions, at least in the smooth case, may be characterized by the differential inequality $\Delta h\geqq 0$, there arises the question as to whether it is possible to characterize the subharmonic functions by a minimizing property with respect to unilateral variations. Subject to reasonable regularity conditions, the author finds that a subharmonic function $u(x, y)$ may be characterized by the property of minimizing the Dirichlet integral $I(u)$ with respect to all admissible functions $v(x, y)$ such that $v(x, y)=u(x, y)$ on the boundary, and $v(x, y)\leqq u(x, y)$ in the interior of the region of integration. In fact, the author first develops a theory of unilateral variations, for nonparametric double integral problems in the calculus of variations, that is sufficiently general to serve as a basis for applications to subharmonic functions, as well as to the Plateau problem in nonparametric form. Approximations by integral means are extensively used in the paper.

T. Radó (Columbus, Ohio).

†**Carson, Albert B. An analogue of Green's theorem for multiple integral problems in the calculus of variations.** Contributions to the Calculus of Variations, 1938–1941, pp. 453–489. University of Chicago Press, Chicago, Ill., 1942. $3.00.

The central problem of this paper is the determination of necessary and sufficient conditions that

$$(1)\quad \int_A [v_\alpha(x)\partial\eta/\partial x_\alpha + w(x)\eta(x)]dA = \int_B \eta(x)v_\alpha(x)l_\alpha dB$$

for every function η in a suitably chosen class H of admissible functions. Here A is a region of n-dimensional x-space in which the functions w, v_α and η are defined, B denotes its boundary and the l_α are the direction cosines of the outer normal of the boundary B. Previous derivations of such conditions have usually been based on Green's formula. The present proof is made independently of Green's formula except for the trivial case when the region A is a cube, with edges parallel to the coordinate axes. The functions v_α are supposed to be continuous on $A+B$ and the function w is supposed to be summable on A. In three-space the conditions on the boundary B are that it be composed of a finite number of nonsingular surfaces of class C' which intersect each other in a finite number of rectifiable curves. In n-space the statement of the conditions on B is slightly more complicated. Then a necessary and sufficient condition for (1) to hold is:

$$(2)\quad \int_{A'} [w(x)dA = \int_{B'} v_\alpha(x)l_\alpha dB$$

for every cube A' with edges parallel to the axes. A modified form of the condition is obtained for the case when the functions v_α have discontinuities. The proof of the equivalence of (1) and (2) in the case when the v_α are continuous shows that the equality may be replaced by "$\geqq$" in both places provided $\eta(x)\geqq 0$ in (1). When the functions v_α are absolutely continuous in the sense of Tonelli, the condition (2) is equivalent to the equation $w=\partial v_\alpha/\partial x_\alpha$ almost everywhere. As a corollary Green's formula is obtained under rather general hypotheses. Applications are given to the obtaining of necessary conditions in multiple integral problems of the calculus of variations, including the Haar-Coral equations, ridge or edge conditions, and an analogue of the Jacobi condition in the Haar form. The problem of one-sided variations is also considered. A number of other formulas are obtained by the author as corollaries of his theorems. *L. M. Graves* (Chicago, Ill.).

McShane, E. J. On the theory of relative extrema. Revista Ci., Lima **43**, 111–134, 475–482, 659–666 (1941); **44**, 85–92 (1942). (Spanish) [MF 4844]

This paper is concerned with necessary conditions for a value $z=z_0$ to render a given function $f^0(z)$ an extremum in the class of values z belonging to a set Z and satisfying equations $f^1(z)=0=\cdots=f^p(z)$. The purpose of this discussion is two-fold: first, to establish conditions of sufficient generality to be applicable in the case of strong relative extrema in the calculus of variations; secondly, in the particular case of $p=1$, to obtain second-order conditions without normality assumptions. The two general theorems established are independent of any restrictions placed specifically on the character of the variable z and its range Z, or on the character of the individual functions $f^i(z)$ $(i=0, 1, \cdots, p)$. Instead, the following general hypothesis is made: there exist two sets V and W of vectors $(u^0, \cdots, u^p)$ in $(p+1)$-dimensional space such that, if $v_j=(v_j^i)$ $(i=0, 1, \cdots, p;\ j=1, \cdots, n)$ are n vectors of V, and $w_k=(w_k^i)$ $(i=0, 1, \cdots, p;\ k=1, \cdots, m)$ are m vectors of W, then there exists a function

$$z(b, e)=z(b_1, \cdots, b_n;\ e_1, \cdots, e_m)$$

defined on an interval $0\leqq b_j\leqq\epsilon$, $-\epsilon\leqq e_k\leqq\epsilon$ $(\epsilon>0;\ j=1, \cdots, n;\ k=1, \cdots, m)$, whose values are in Z, which reduces to z_0 for $b_j=0=e_k$, and such that the functions $f^i(z(b, e))$ $(i=0, 1, \cdots, p)$ are of class C'' and satisfy

$$f^i_{b_j}(z(0, 0))=v_j^i \qquad (j=1, \cdots, n),$$
$$f^i_{e_k}(z(0, 0))=0,$$
$$f^i_{e_k b_j}(z(0, 0))=w_k^i \quad (\alpha=1, \cdots, p;\ i=0, 1, \cdots, p;\ k=1, \cdots, m).$$

Relative to such vector spaces V and W, which correspond, respectively, to the usual first and second variations of the functions $f^i(z)$, the author establishes the following general theorems: (I) if $z=z_0$ renders a minimum to $f^0(z)$ in the class of values z of Z satisfying $f^\alpha(z)=0$ $(\alpha=1, \cdots, p)$, then there exist constants $\gamma_0\geqq 0, \gamma_1, \cdots, \gamma_p$ not all zero and such that $\gamma_0 v^0+\cdots+\gamma_p v^p\geqq 0$ for all vectors v of V; (II) if $z=z_0$ renders a minimum to $f^0(z)$ in the class of values z of Z satisfying a single auxiliary equation $f^1(z)=0$, then there exist two constants $\gamma_0\geqq 0$, γ_1 not both zero and such that $\gamma_0 u^0+\gamma_1 u^1\geqq 0$ for all vectors $u=(u^0, u^1)$ which belong to either V or W. These theorems are deduced from the properties, in particular the convexity, of certain associated sets in $(u^0, \cdots, u^p)$-space. Finally, the author states without proof that the above result (II) may be extended to the case $p>1$ whenever the set V contains all vectors $u\equiv(u^0, \cdots, u^p)$ which satisfy equations of the form

$$u^0+A_1u^1+\cdots+A_pu^p=0, \quad B_1u^1+\cdots+B_pu^p=0.$$

The most important given application of these results is to the case of the isoperimetric problem of the calculus of variations in parametric form, and with fixed end-points. As far as the calculus of variations is concerned, however, the author has already published these results in the more general setting of the Lagrange problem [Amer. J. Math. **61**, 809–819 (1939) and **63**, 516–530 (1941); these Rev. **1**, 78 and **3**, 53]. *W. T. Reid* (Chicago, Ill.).

McShane, E. J. Sufficient conditions for a weak relative minimum in the problem of Bolza. Trans. Amer. Math. Soc. **52**, 344–379 (1942). [MF 7123]

The present paper is concerned with sufficient conditions for a weak relative minimum for the problem of Bolza in the calculus of variations. Both the parametric and non-parametric cases are considered, theorems for the latter being deduced from those of the former. In the earlier sufficiency theorems, it was assumed that the arc E under consideration satisfies with a set of multipliers a set of conditions which are shown to imply that E is a minimizing arc. In the present paper it is shown that it is sufficient to assume that these multipliers are functions of the variations under consideration, whereas heretofore it was assumed that they were independent of these variations. This weakening of hypotheses apparently makes it necessary to devise a sufficiency proof that is independent of field theory. The proof used is an indirect one and, as one would expect, is more complicated than the corresponding proof based on more stringent hypotheses and using the field theory. It is shown by an example that the results given here are appli-

cable to cases not covered by any previous published theorem. *M. R. Hestenes* (Chicago, Ill.).

Pâquet, P. V. Les formes différentielles extérieures Ω_n dans le calcul des variations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 65–84 (1941). [MF 6902]

A consideration of the multiple integral

$$I_n=\int_D F(x^i, y^\alpha(x), y_i^\alpha(x))dx^1\cdots dx^n$$

over the domain D of the variables $x^1, \cdots, x^n$. Several theorems are given on differential forms of minimum rank which arise in connection with the variation problem for the above integral. *T. Y. Thomas* (Los Angeles, Calif.).

Pâquet, P. V. Sur la géométrie différentielle suivant la méthode de Grassmann et les intégrales du calcul des variations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 148–168 (1941). [MF 6903]

This begins with a consideration of the integral

$$\int_D F(t^i, x^\mu, \partial x^\mu/\partial t^i)dt^1\cdots dt^n,$$

with $i=1, 2, \cdots, n$ and $\mu=1, 2, \cdots, N$ $(N>n)$ and a statement of the conditions on the integrand for invariance of this integral. The N dimensional space E_N and its n dimensional subspace Σ_n are taken up, tangent spaces introduced and the transformation of vectors discussed. The dual tangent space is brought in, also tensor densities associated with Σ_n. The paper is formal and clearly written. It closes with a remark on the intrinsic form of integrals in the calculus of variations. *T. Y. Thomas.*

Mickle, Earl J. Associated double integral variation problems. Duke Math. J. **9**, 208–227 (1942). [MF 6349]

The present paper is concerned with double integral variation problems associated with the problem of minimizing a double integral

$$J(z)=\int\int_R F(p, q)dxdy, \quad p=\partial z/\partial x; \quad q=\partial z/\partial y,$$

and with corresponding problems in the parametric case. Let $X=-F_p$, $Y=F_q$, $Z=F-pF_p-qF_q$, $\Delta=F_{pp}F_{qq}-F_{pq}^2$. By a suitable transformation expressing new variables $\bar{p}$, $\bar{q}$, $\bar{F}$, $\bar{\Delta}$ in terms of X, Y, Z, p, q, F and $\bar{x}$, $\bar{y}$, $\bar{z}(\bar{x}, \bar{y})$ in terms of x, y, z, p, q, one obtains a variational problem in the new variable with the property that extremals are transformed into extremals. The author studies twenty-four transformations of this type and shows that they form a group. The results are applied to the Dirichlet and the area integral. In the parametric case similar results are obtained for a set of eight transformations forming a group. *M. R. Hestenes* (Chicago, Ill.).

Young, L. C. Generalized surfaces in the calculus of variations. II. Ann. of Math. (2) **43**, 530–544 (1942). [MF 7008]

[The first part appeared in Ann. of Math. (2) **43**, 84–103 (1942); these Rev. **3**, 249.]

The author considers the double integral problem

$$\int\int f(x, y, p, q)dxdy=\text{minimum}$$

under surprisingly weak assumptions concerning the integrand $f(x, y, p, q)$ (which do not even require f to be continuous, but merely measurable B!). It is shown that the lower bound is unaltered if one admits, in addition to ordinary surfaces, special kinds of generalized surfaces called "mean surfaces." With this result, it is easy to establish convexity properties of the minimum in the class of ordinary Lipschitzian surfaces as well as in the class of generalized Lipschitzian surfaces. It is then shown that a surface is a solution of the minimum problem (either in the ordinary or in the generalized cases) if and only if a certain inequality (W, E) is satisfied. This inequality is a generalization of the Weierstrass condition and the Euler equation combined, and is equivalent to them in case f satisfies the usual differentiability assumptions. The inequality (W, E) and its proof are intimately related to the theorem establishing the existence of a flat support at each point of a convex body. *M. Shiffman* (New York, N. Y.).

Lonseth, A. T. The problem of Plateau in hyperbolic space. Amer. J. Math. **64**, 229–259 (1942). [MF 6429]

The author considers the hyperbolic space of constant negative curvature -1, the model being the upper half-space $z>0$ with line element

$$ds^2=(dx^2+dy^2+dz^2)z^{-2},$$

and solves the problem of finding a hyperbolic minimal surface of class C'' bounded by a given Jordan curve. The essential difficulty, beyond the methods used in the ordinary (Euclidean) Plateau problem, is the unavailability of a theory analogous in the Euclidean case to potential theory. In its stead, use is made of classic theorems concerning solutions of elliptic differential equations and some theorems of Morrey concerning the growth of Dirichlet integrals. With these theorems the author shows how to construct a minimizing sequence with uniformly bounded derivatives up to the third order in any closed interior domain. The existence of a solution of class C'' then follows. The question of the analyticity of the solution is not discussed. *M. Shiffman* (New York, N. Y.).

GEOMETRY

*Levi, F. W. **Finite Geometrical Systems.** University of Calcutta, Calcutta, 1942. iii+51 pp.

This course of six lectures describes the manner in which various branches of mathematics, namely the theory of groups, of fields, of hypercomplex numbers and topology, all contribute to the construction of finite geometrical systems, which have lately been raised to serious status by their unexpected application to statistics. The first lecture begins with a historical sketch and then deals with applications of the theorem that every even regular graph of degree m is the product of m regular graphs of degree 1. A certain class of such graphs can be regarded as balanced incomplete block designs [R. C. Bose, Ann. Eugenics **9**, 353–399 (1939); these Rev. **1**, 199] of the special kind known as symmetric (that is, with $r=k$ and therefore $b=v$).

The second lecture provides a lucid account of regular divisions of a surface into α_2 cells by means of α_1 segments connecting α_0 points. The limitations of this theory in application to combinatorics are illustrated by a fallacious "proof" that Euler's problem of the 36 officers has no solution. However, block designs with $v=r+1$, $b=\frac{1}{2}r(r+1)$, $k=3$, $\lambda=2$ are related to the problem of dividing a surface

of characteristic $-\alpha_0+\alpha_1-\alpha_2=(r+1)(\frac{1}{2}r-1)$ into $r+1$ mutually neighboring cells. [This has been solved for $r=2, 3, 5, 6, 8, 9, 11$. See Bose, loc. cit., pp. 373–380.]

The third lecture deals with Galois fields and finite geometries; details are given for the example $PG(2, 2^s)$. It is shown that the nondegenerate conics of $PG(2, t)$, where $t=p^n$, form a block design with $v=t^2+t+1$, $r=t^2(t^2-1)$, $b=vt^2(t-1)$, $k=t+1$, $\lambda=t^2(t-1)$, and that the parabolas of $EG(2, t)$ form a block design with $v=t^2$, $r=t(t^2-1)$, $b=rt$, $k=t$, $\lambda=t(t-1)$.

The fourth lecture shows how Desargues' theorem provides a system of coordinates belonging to a division ring (or "skew field"), and then gives Witt's proof of Wedderburn's theorem (that every finite division ring is a field), from which it follows that all finite Desarguesian geometries are generated by Galois fields and that all finite projective geometries not generated by Galois fields are two-dimensional. The existence of non-Desarguesian geometries is established by an instance due to Tschetweruchin.

The fifth lecture solves the problem of constructing a finite non-Desarguesian geometry (for which Tschetweruchin's method is inadequate). This is done by means of coordinates belonging to a linear algebra which is not a ring as it lacks one of the distributive laws. The collineations of such a geometry (with $t+1$ points on each line) form a doubly transitive group of degree t and order $t(t-1)$; therefore t is a power of a prime. Conversely, every doubly transitive group of degree p^n and order $p^n(p^n-1)$ provides a plane geometry, and this geometry is non-Desarguesian provided the subgroup which leaves invariant one particular object is non-Abelian. An actual instance is given with $p^n=3^2$. Incidentally, it is pointed out that every finite plane geometry leads to an Eulerian (or Greco-Latin) square, whence the impossibility of Euler's arrangement of officers (satisfactorily proved by Tarry) implies the impossibility of a geometry with $t=6$. Other cases, such as $t=10$, remain undecided; we can only say that every known finite plane projective geometry has p^n+1 points on each line.

The sixth lecture describes Ruth Moufang's results on non-Desarguesian geometries which admit a unique fourth harmonic point (for any three collinear points), and then gives a proof that such geometries cannot be finite. [This is the deepest part of the book.] The geometrical results may be summarized as follows. Of the three famous theorems, Pappus', Desargues' and the unique harmonic point, every finite geometry admits either all or none. The former case is $PG(N, p^n)$; the latter is necessarily two-dimensional.

H. S. M. Coxeter (Toronto, Ont.).

*Coxeter, H. S. M. **Non-Euclidean Geometry.** Mathematical Expositions, no. 2. University of Toronto Press, Toronto, Ont., 1942. xv+281 pp. $3.25.

This book presents a very readable account of the fundamental principles of hyperbolic and elliptic geometries. The approach is by way of projective geometry, the three chapters immediately following a brief historical sketch being devoted to an excellent survey of the foundations of real projective geometry. From the axioms [Pieri, Vailati, Dedekind] through the introduction of homogeneous coordinates [von Staudt-Hessenberg] the treatment is necessarily sketchy since this background material is doubtless intended merely to set the stage for what follows. Chapters V, VI, VII are devoted to elliptic geometry of one, two and three dimensions. Observing that the axioms of real projective geometry are valid in elliptic geometry, the latter is obtained by choosing an absolute polarity. Preparatory to the study of hyperbolic geometry, the author discusses a "descriptive geometry" (not to be confused with the subject ordinarily so labeled), the axioms [Veblen] of which are in terms of point and the order relation [ABC]. (The geometry of a convex region is a model.) Instead of obtaining the hyperbolic metric by means of an absolute polarity in descriptive space (extended by the introduction of ideal elements), the author prefers to introduce the notion of congruence (five axioms) and obtain an "absolute geometry" which bifurcates into Euclidean and hyperbolic geometry upon the adjunction of the Euclidean or hyperbolic axiom of parallelism. The concluding chapters are concerned with circles and triangles, use of a general triangle of reference, area and Euclidean models.

The book contains a wealth of material of the classic and modernized-classic type. Though perhaps beyond the scope of the author's plan, the book would have been considerably enriched, in the reviewer's opinion, by an appendix surveying the most recent contributions to non-Euclidean geometries. These include, for example, the highly interesting foundations of hyperbolic geometry based upon the sole operations of joining and intersecting due to Menger, Jenks, and Abbott [Jenks, Rep. Math. Colloquium (2) **2**, 10–14 (1940); ibid. (2) **3**, 3–12 (1941); Abbott, Rep. Math. Colloquium (2) **3**, 13–27 (1941); Menger, Rice Inst. Pamphlet **27**, 41–79 (1940); these Rev. **3**, 180–181] as well as the purely metric foundations given by the reviewer [Revista Ci., Lima **40**, 3–20 (1938)]. *L. M. Blumenthal.*

Greenwood, Thomas. Les fondements de la géométrie euclidienne. Rev. Trimest. Canad. **28**, 195–223 (1942). [MF 6957]

The author gives a foundation for three-dimensional Euclidean geometry in terms of point and distance in which the properties of the Euclidean line are determined by a single postulate (postulate of zero curvature) imposed upon a set of nine so-called generic postulates which apply equally well to non-Euclidean lines. Postulates of order, congruence and continuity follow the Hilbert pattern. Defining coplanar lines to be parallel when they are perpendicular to the same straight line, the theory of parallels requires no new postulate, being developed upon the Euclidean character of the line assured by the previous assumptions. The author makes no reference to other metric foundations of geometry [e.g., Menger, Busemann, Birkhoff, Gillam and the reviewer].

L. M. Blumenthal (Columbia, Mo.).

Silberstein, Ludwik. The space relation among five points in elliptic and hyperbolic geometry. Philos. Mag. (7) **33**, 536–540 (1942). [MF 7090]

After exhibiting the Cayley determinant relation for five points in Euclidean three space, the author remarks that to the best of his knowledge "the relation for a non-Euclidean space has not been considered," and the object of the note is to derive five-point relations for elliptic and hyperbolic spaces. The author succeeds in obtaining the well-known results deduced by Schering [Nachr. Ges. Wiss. Göttingen **1870**, 311–321], Mansion [Ann. Soc. Sci. Bruxelles. Mem. **19**, 189–196 (1895)] and others, his method not differing essentially from that of Mansion. It should be remarked that, in the present note as well as in the earlier papers, the five-point relation found for "elliptic space" is, more properly, the five-point relation for spherical space. Thus, for example, an elliptic line $E_{1,2}$ contains three points p_1, p_2, p_3

with $\cos(p_ip_j/\mu)=1/2$, $(i,j=1,2,3)$, but the determinant $|\cos(p_ip_j/\mu)|$ $(i,j=1,2,3)$ of these three points does not vanish. *L. M. Blumenthal* (Columbia, Mo.).

Fraile, V., Fraile, A. and Crespo, C. Locus and loci of point domains in the plane. Union Mat. Argentina, Publ. no. 23, 46 pp. (1942). (Spanish) [MF 6888]

Apart from two additions, this paper is a reprint of an earlier one [Revista Union Mat. Argentina 7, 45–50, 87–91, 114–119, 144–169 (1941); cf. these Rev. 2, 294; 3, 251]. *P. Scherk* (Bloomington, Ind.).

Neumann, B. H. Some remarks on polygons. J. London Math. Soc. 16, 230–245 (1941). [MF 6618]

The first part establishes again some of the results of J. Douglas [J. Math. Phys. Mass. Inst. Tech. 19, 93–130 (1940); these Rev. 1, 261]. [The author states that the present paper was in print for Compositio Mathematica in 1940, but was not published.] In the second part certain inequalities due to Finsler and Weitzenböck are generalized. Let $a_{n+1}=a_1, \cdots, a_n$ be the vertices of an n-gon in the Euclidean plane $S=\sum\overline{a_ia_{i+1}}^2$, x any point, c the centroid $B_n=\sum\overline{xa_i}^2$ of $a_1, \cdots, a_n$ and finally $f^2_{\mu,\nu}$ the (algebraic) area of the oriented triangle $xa_\mu a_\nu$. Then $B_n\geqq 16\sum_{\mu<\nu}(f^2_{\mu,\nu})^2$, $S\geqq 2(1-\cos(2\pi/n))B_c$ and $B_c>(2/\sin(2\pi m_1/n))\cdot|\sum f^2_{i,i+1}|$, where $m_1=[(m+1)/4]$. *H. Busemann* (Chicago, Ill.).

Mitra, Abha. On affine regularity of polygons. Bull. Calcutta Math. Soc. 33, 49–55 (1941). [MF 6169]

Using J. Douglas' terminology [see the preceding review] an n-gon in the complex plane is called affine regular of the ω^p-type if its vertices can be obtained from $(1, \omega^p, \omega^{2p}, \cdots, \omega^{(n-1)p})$, $\omega=e^{2\pi i/n}$, $p=1, 2, \cdots, n-1$, by an affinity. A polygon $a_1, \cdots, a_n$, $a_{n+i}=a_i$ is affine regular if and only if the following three conditions are satisfied: (1) no three vertices are collinear; (2) each side is parallel to some diagonal of the form a_ia_{i+s}; (3) each diagonal of the form a_ia_{i+2} is parallel to a diagonal of the form a_ia_{i+4}. *H. Busemann.*

Court, N. A. On the theory of the tetrahedron. Bull. Amer. Math. Soc. 48, 583–589 (1942). [MF 7054]

Corresponding to the nine-point circle of a triangle, there are two twelve-point spheres of an orthocentric tetrahedron: the "first" contains the midpoints of the edges and the "second" the centroids of the faces. The latter can still be defined for the general tetrahedron, and is then known simply as "the twelve-point sphere." This paper is concerned with two new spheres, either of which will invert the twelve-point sphere into the circumsphere. The author names these the G-sphere and the quasi-polar sphere; their centers are the centroid and the Monge point. (The quasi-polar sphere is imaginary if the Monge point is interior to the circumsphere.) Their properties provide generalizations for many known theorems on orthocentric tetrahedra. In fact, when the tetrahedron is orthocentric, the G-sphere is the "first" twelve-point sphere, and the quasi-polar sphere is the polar sphere. *H. S. M. Coxeter* (Toronto, Ont.).

Bioche, Ch. Sur des octogones et des décagones à côtés opposés parallèles. Enseignement Math. 38, 323–324 (1942). [MF 7306]

Loong, Chi-Ho. Some analogs of the triangle geometry in the Kasner plane. Nat. Math. Mag. 17, 8–12 (1942). [MF 7235]

Some new theorems on triangles are added to those given by J. De Cicco [Amer. Math. Monthly 46, 627–634 (1939); these Rev. 1, 169]. There is a definition of a generalized orthocenter, called horn center, and others of a generalized orthopole of a line and of the Simpson line. The theory is not only invariant under the metric G_3 of Kasner, but also under his similitude group G_4. *D. J. Struik.*

Peiser, A. M. The Hagge circle of a triangle. Amer. Math. Monthly 49, 524–527 (1942). [MF 7344]

Thébault, V. Sur le théorème de Pilatte. Enseignement Math. 38, 287–293 (1942). [MF 7304]

Labra, Manuel. A generalization of Stewart's theorem. Application to the computation of the sides of important pedal triangles. Revista Soc. Cubana Ci. Fís. Mat. 1, 13–18 (1942). (Spanish) [MF 7192]

Algebraic Geometry

Weiss, E. A. Die Koppelkurve als Laguerresches Bild einer Hesseschen Korrespondenz. Math. Z. 47, 187–198 (1941). [MF 6818]

G. T. Bennett [Proc. London Math. Soc. (2) 20, 59–84 (1922)] obtained the characteristics of the couple curve discussed in this paper (called the three-bar sextic by Bennett) as the locus of the Laguerre images of the conjugate points on the Hessian of an elliptic cubic, thus treating a curve defined in the field of kinematics by the methods of algebraic geometry. The three basis points of the involution on the Hessian are double points on the elliptic tricircular sextic couple curve.

In the present paper, the author, using the above methods, determines the following types of couple curves, each associated with a similar type of cubic. An elliptic cubic determines an elliptic couple curve with one or two branches according as the cubic has one or two branches. A rational cubic determines a rational couple curve. In this case the couple curve has an additional double point at the double point of the cubic. A reducible cubic determines a reducible couple curve. In particular, if the cubic consists of a conic and a line, the couple curve consists of a bicircular quartic and a circle; if three lines, the couple curve consists of three circles. *T. R. Hollcroft* (Aurora, N. Y.).

Wade, T. L. Euclidean concomitants of the ternary cubic. Bull. Amer. Math. Soc. 48, 589–601 (1942). [MF 7055]

The Euclidean concomitants of the ternary cubic $l_{ijk}x^ix^jx^k$ can all be expressed as tensors of rank zero formed by combining, according to the ordinary rules of tensor algebra, the tensors l_{ijk}, x^i, u_i, ϵ^{ijk}, L_i, E^{ij}, where u_i is the vector contravariant to x^i, ϵ^{ijk} is the numerical tensor usually so denoted, L_i is the vector $[0, 0, 1]$ and

$$E^{ij}=\begin{bmatrix}1 & \cdot & \cdot\\ \cdot & 1 & \cdot\\ \cdot & \cdot & \cdot\end{bmatrix}.$$

Using this fact, an irreducible system of Euclidean invariants of the ternary cubic, complete through the fifth degree, is constructed, and also an irreducible system of covariants, complete through the fourth degree. Numerous geometrical interpretations are given. *W. V. D. Hodge.*

Bronowski, J. On triple planes. III. J. London Math. Soc. 17, 80–87 (1942). [MF 7315]

Extending the ideas of an earlier paper [J. London Math. Soc. 17, 24–31 (1942); these Rev. 3, 303] the author considers a surface M' containing an m-fold involution which has a model s. A transformation $f' \to f_1'$ of curves on M' is defined by the condition that $f'+f_1'$ be the complete image of a curve on s. This transformation is linear and has the property that $f'+f_1' \to (m-1)(f'+f_1')$. It follows that $f' \to kf'$ only if $k=m-1$ or -1; and a base, with division, is constructed for the curves on M' of the type $c^{(1)'}, \cdots, c^{(t)'}, d^{(t+1)'}, \cdots, d^{(s)'}$, where $c^{(i)'}c^{(j)'}=0=d^{(i)'}d^{(j)'}$ if $i \neq j$, $c^{(i)'}d^{(j)'}=0$, and $c^{(i)'} \to (m-1)c^{(i)'}$, $d^{(i)'} \to -d^{(i)'}$; t is the number of curves in a base of s. *R. J. Walker.*

Muhly, H. T. Valuations and infinitely near algebraic loci. Amer. J. Math. 64, 457–487 (1942). [MF 6943]

In this paper the arithmetic theory of infinitely near points in the plane, developed by the reviewer in the case of algebraically closed ground fields of characteristic zero, is generalized to arbitrary ground fields K of characteristic zero. The generalization is carried out by extending the polynomial ring $K[x, y]$ to $K'[x, y]$, where K' is the least normal extension of K which contains the residue field of the valuation B (or branch) under consideration. The main lemma is to the effect that any simple v-ideal of B in $K[x, y]$, of kind h, factors in $K'[x, y]$ into a product of simple v-ideals of the same kind h, belonging to the various (conjugate) extensions of B. By virtue of this result, the properties of v-ideals, valid in $K'[x, y]$, especially the unique factorization theorem for simple ideals, remains valid in $K[x, y]$. The geometric significance of this generalization derives from the possibility of regarding any simple irreducible V_{r-s} on a V_r as a simple point of a surface over a suitable transcendental extension of the ground field. Therefore the theory provides a means for the analysis of the neighborhood of a V_{r-s} on a V_r and for the formal definition of the concept of V_{r-s}'s which are infinitely near a given V_{r-s}. *O. Zariski* (Baltimore, Md.).

Zariski, Oscar. A simplified proof for the resolution of singularities of an algebraic surface. Ann. of Math. (2) 43, 583–593 (1942). [MF 7013]

This paper simplifies the arithmetical proof given by the author [Ann. of Math. (2) 40, 639–689 (1939); these Rev. 1, 26; referred to as "Reduction"] of the theorem that an algebraic surface over a ground field K of characteristic zero can be birationally transformed into one without singular points. In the earlier proof it was essential that K should be algebraically closed; this condition is no longer required. Broadly speaking, the simplification consists in the elimination of the intricate argument necessary to show that the isolated singularities of a normal surface can be removed, and using instead a much simpler process involving "resolving systems."

The theorem of local uniformization, which states that, given any valuation of an algebraic function field Σ/K, there exists a projective model V of Σ on which the center of the valuation is a simple subvariety of V, is taken from an earlier paper by the author [Amer. J. Math. 62, 187–221 (1940); these Rev. 1, 102]. After obtaining some preliminary results on birational correspondences it is shown that the essential results in the theory of quadratic transformations obtained in "Reduction" can be extended to the case in which the ground field K is not algebraically closed. An arithmetical proof is also given of the theorem that a quadratic transformation with simple center P on a surface F transforms F into a surface F' on which the variety $T(P)$, corresponding to P, is such that every point of it is a simple point of F'.

If W is any set of places on the surface F, a set of surfaces $F_1, F_2, \cdots, F_n$, birationally equivalent to F, is called a resolving system of W if every zero-dimensional valuation of Σ/K (Σ the function field of F) having center on W has its center at a simple point of at least one of the surfaces F_i. The existence of a finite resolving system for any set W is proved by using the theorem ["Reduction," p. 681] that the limit ring of a normal sequence of quotient rings in Σ is the valuation ring of a zero-dimensional valuation of Σ/K. Clearly F_{n-1}, F_n is the resolving system of a subset W' of W. Quadratic transformations are used to eliminate fundamental points of the birational transformation which exists between F_{n-1} and F_n, and two new surfaces F_{n-1}, F_n are obtained which form a resolving system for W'. It is then shown that the join F_{n-1}^* of F_{n-1}' and F_n' (that is, in the case in which K is algebraically closed, the surface of pairs of corresponding points of F_{n-1}' and F_n') is a resolving system for W'. Then $F_1, F_2, \cdots, F_{n-2}, F_{n-1}^*$ is a resolving system for W. Repeating the argument $n-1$ times, and taking $W=F$, we arrive at a resolving system for F consisting of a single surface, and this proves the theorem. *W. V. D. Hodge* (Cambridge, England).

Abellanas, P. F. Dimension of an algebraic variety. Revista Mat. Hisp.-Amer. (4) 2, 13–21 (1942). (Spanish) [MF 7081]

The algebraic varieties of points associated with the ideals of polynomials in n unknowns over a field are the elements of a structure in the sense of Ore [Ann. of Math. (2) 36, 406–437 (1935)]. A unique dimension is assigned to each variety by means of a descending principal chain. *J. L. Dorroh* (Baton Rouge, La.).

Hodge, W. V. D. The intersection formulae for a Grassmannian variety. J. London Math. Soc. 17, 48–64 (1942). [MF 6974]

This paper is a continuation of a recent paper by the author [J. London Math. Soc. 16, 245–255 (1941); these Rev. 3, 304], in which it was proved that the Schubert varieties of a given dimension on a Grassmannian variety $\Omega_{n,p}$ form a base for the algebraic subvarieties of $\Omega_{n,p}$ of that dimension (in the sense of algebraic equivalence). To find the intersection of two varieties on $\Omega_{n,p}$, in the sense of equivalence, it is therefore sufficient to know how to express the intersection of two Schubert varieties in terms of the Schubert varieties of the base. This problem is solved in the following fashion. Let $\Omega_{a_0,a_1,\cdots,a_p}$ be any Schubert variety and let Σ_h be the special Schubert variety which represents the p-spaces incident with a given S_{n-p-h}. The following equivalence due to Pieri is proved:

$$\Omega_{a_0,a_1,\cdots,a_p} \cdot \Sigma_h = \sum \Omega_{b_0,b_1,\cdots,b_p},$$

where the summation is over all values of $b_0, b_1, \cdots, b_p$ such that $a_{j-1} < b_j \leqq a_j$ $(a_{-1}=0)$, $\sum_0^p b_i = \sum_0^p a_i - h$. This special intersection formula leads to the solution of the general problem, in view of the following fact: it is proved that every Schubert variety $\Omega_{a_0,a_1,\cdots,a_p}$ can be put in the form of a determinant $|\Delta_{i,j}|$, where $\Delta_{i,j}=\sum_{n-p-a_i+j}$, $i, j=0, 1, \cdots, p$, and where products of Δ's are to be regarded as intersections. *O. Zariski* (Baltimore, Md.).

Differential Geometry

Brauner, K. und Müller, Hans Robert. Über Kurven, welche von den Endpunkten einer bewegten Strecke mit konstanter Geschwindigkeit durchlaufen werden. Math. Z. 47, 291–317 (1941). [MF 6824]

From the introduction: ". . . liefern die Untersuchungen der erwähnten Bewegungen neue Kennzeichnungen von Geraden, Kreisen und Zindlerschen Kurven, im Raume ergeben sich interessante Eigenschaften der Striktionslinie von windschiefen Regelflächen und der Gratlinie von Torsen. Es gelingt so eine neue Definition der Striktionslinie im Reellen zu geben, bei der auf jede Limesbetrachtung verzichtet werden kann. Fast zwangsläufig gelangt man zu dem geometrisch interessanten Begriff der Verwindung von Regelflächen. Einige Beispiele und eine Kennzeichnung der Schraublinie unter den Raumkurven beschliessen die Arbeit." *P. Scherk* (Bloomington, Ind.).

Bagchi, Haridas. Cylindroid and kindred surfaces. Bull. Calcutta Math. Soc. 33, 129–146 (1941). [MF 7028]

The purpose of this paper is to investigate the cylindroid and a few of its generalizations in a pure mathematical way without referring to its mechanical properties. The chief topics of the paper are: the relation of the cylindroid to the plane at infinity and the absolute circle; the examination of the surfaces which have, like the cylindroid, a rectilinear line of striction; and the investigation of polar quadrics and of the loci of points whose polar quadrics with respect to the cylindroid are specialized, for example, surfaces of revolution or paraboloids. *E. Helly.*

Gheorghiu, Gh. Th. Sur une classe de surfaces. Mathematica, Timişoara 18, 55–67 (1942). [MF 7214]

This paper follows one of the same name published earlier by the author [Bull. Sci. École Polytéch. Timişoara 10, 87–92 (1941)]. Both deal with a class of surfaces (S), previously studied by Tzitzeica and O. Mayer, defined centroaffinely by $K/d^4=$const. along a family of asymptotic lines ($K=$curvature, $d=$distance from origin to tangent plane). From the special form which the centroaffine differential equations of definition of any surface take for the surfaces (S) several of their geometric properties are derived. For example, they are completely characterized by a special position of the centers of their Lie quadrics. The second part of the paper makes strong use of a special type of transformation of surfaces devised by M. Jonas in 1920 in connection with W-congruences. By this means it is shown how any surface (S) can be transformed into ∞^4 surfaces of the same class, the defining asymptotic families corresponding. This investigation also gives rise to an analogue for surfaces (S) of Bianchi's theorem of permutability for pseudospheres. *J. L. Vanderslice* (College Park, Md.).

Blanc, Ch. et Fiala, F. Le type d'une surface et sa courbure totale. Comment. Math. Helv. 14, 230–233 (1942).

A Riemann plane is defined as a plane of rectangular coordinates (x, y) with a metric defined by

$$ds^2=Edx^2+2Fdxdy+Gdy^2,$$

where E, F, G are analytic functions, and such that the length of every divergent line is infinite. According to the fundamental theorem of the theory of uniformization, such a plane can be mapped conformally either on the interior of the unit circle (hyperbolic type) or on the punctured plane (parabolic type). The following criterion for type is obtained: if the integral of the total curvature of a Riemann plane is bounded from below, then the plane is of parabolic type. The proof follows that given by Ch. Blanc [Comment. Math. Helv. 13, 54–67 (1940); these Rev. 2, 293] for the corresponding theorem concerning Riemann nets of discrete points, the transition to surfaces being based on the notion of "true circle" developed by F. Fiala [Comment. Math. Helv. 13, 293–346; these Rev. 3, 301].

E. F. Beckenbach (Ann Arbor, Mich.).

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XXXIII. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 26, 135–177 (1939). [MF 7198]

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XXXIV. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 26, 179–231 (1939). [MF 7199]

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XXXV. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 26, 233–287 (1939). [MF 7200]

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XXXVI. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 26, 289–341 (1939). [MF 7201]

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XXXVII. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 28, 1–165 (1940). [MF 7202]

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XXXVIII. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 28, 167–258 (1940). [MF 7203]

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XXXIX. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 28, 299–321 (1940). [MF 7205]

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XL. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 29, 1–12 (1940). [MF 7207]

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XLI. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 29, 37–58 (1941). [MF 7209]

These are new installments of a series started by the author in 1932. In each paper the author treats several mutually unrelated topics (for example, special systems of curves) mostly belonging to the fields of conformal and inversive geometry. [The author's name has formerly been S. Nakajima, sometimes spelled Nakazima, and also Matsumura.] *W. Feller* (Providence, R. I.).

Grove, V. G. The transformation T of congruences. Ann. of Math. (2) 43, 623–633 (1942). [MF 7395]

Continuing investigations of Fubini and Cook, the author studies transformations T of two rectilinear congruences Γ and Γ^* with the following properties. (1) The correspondence between the lines of Γ and Γ^* is biunivoque. (2) Corresponding lines are not complanar. (3) The developables of Γ and Γ^* correspond to each other. (4) In each congruence there are at least three transversal surfaces whose tangent planes at their intersection with any line of the congruence pass through the corresponding line of the other congruence. The author finds linear differential equations of first order for the focal points of the two congruences; the independent variables are the parameters u, v of the developables of Γ and Γ^*. It is shown that there are two types of transformations T. (a) Asymptotic transformations. These transform W congruences into W congruences. The lines of a transversal surface corresponding to the developables of one congruence are asymptotic lines. (b) Conjugate transformations. Let x and y be points on two transversal surfaces of Γ^*, and let z, w be the focal points of a line of Γ. The author studies the congruences of the lines (xz), (yw), (yz) and (xw). If three of them

generate W congruences, the fourth will also do so. Furthermore, the author studies some new pencils of congruences deduced from the preceding ones. He finds interesting properties of the asymptotic lines of the transversal surfaces. For both types of transformations, the author shows the connection between them and transformations which have been studied previously. *G. Fubini* (New York, N. Y.).

Botella Raduán, F. Continuity of the components of the fundamental tensor of Riemann space. A class of non-analytic spaces. Revista Mat. Hisp.-Amer. (4) 2, 72–76 (1942). (Spanish) [MF 7131]

The author asserts that he solves a problem concerning Riemannian geometry when only the first derivatives of the g_{ij} are continuous. *A. Weil* (Bethlehem, Pa.).

Ghosh, N. N. Dyadics and multidyadics in hyperspace. Bull. Calcutta Math. Soc. 33, 105–117 (1941). [MF 7026]

The exposition differs from other treatments of the subject mainly in that the author uses his own notations. They are based on the notion of "scalar determinant" and most of them have been introduced in previous papers.

G. Y. Rainich (Ann Arbor, Mich.).

Feld, J. M. Whirl-similitudes, euclidean kinematics, and non-euclidean geometry. Bull. Amer. Math. Soc. 48, 783–790 (1942). [MF 7284]

The author discusses two groups of transformations of plane oriented lineal elements. For the more general group he gives two representations. The lineal elements correspond to plane displacements in the first representation, and to points in a non-Euclidean (quasi-elliptic) three space in the second representation. *P. Franklin.*

Kasner, Edward. Differential equations of the type: $y'''=Gy''+Hy''^2$. Proc. Nat. Acad. Sci. U. S. A. 28, 333–338 (1942). [MF 7069]

This note discusses various problems in geometry and physics which give rise to differential equations of the type $y'''=G(x, y, y')y''+H(x, y, y')y''^2$, and describes some geometric properties of the system of integral curves.

P. Franklin (Cambridge, Mass.).

Terracini, Alejandro. On the differential equation $y'''=G(x, y, y')y''+H(x, y, y')y''^2$. Univ. Nac. Tucumán. Revista A. 2, 245–329 (1941). (Spanish) [MF 6762]

Kasner has developed the geometry of the dynamical trajectories of positional fields of force [Differential-Geometric Aspects of Dynamics, Amer. Math. Soc. Colloquium Publ., vol. 3, part II, New York, 1913 (reprinted 1934)].

The trajectories described by a particle of unit mass in a field of force with components $\phi(x, y)$ and $\psi(x, y)$ are represented by

$$\text{(D)}\quad (\psi-y'\phi)y'''=[\psi_x+y'(\psi_y-\phi_x)-y'^2\phi_y]y''-3\phi y''^2.$$

This is a special form of

$$\text{(G)}\qquad y'''=G(x, y, y')y''+H(x, y, y')y''^2.$$

Kasner gave a complete (metric) characteristic set of five properties of the dynamical type (D). The author completes the problem proposed by Kasner, namely, that of giving a purely projective characterization of the trajectories (D). Moreover, many types of equations of the form (G) are studied in detail. The author's characterization of the trajectories (D) consists of the following five properties each of which is equivalent to the corresponding one of Kasner. (I′) If for each of the ∞^1 trajectories containing a given lineal element E_1 defined by the point A and the direction of the line a through A, the ∞^1 conic sections are constructed which have third order contact with the trajectory at E_1, there results a linear congruence of conics. This means that there is a point P on a and a line p passing through A so that P and p are pole and polar with respect to each of the ∞^2 conics. (II′) The same fixed line p corresponds to each of the ∞^1 lineal elements E_1 through the fixed point A. (III′) As the line a is rotated about the fixed point A, the corresponding point P describes a conic which passes through A and is tangent to the line p at A.

To formulate (IV′) we need some definitions. Denote by q the line tangent at the point P to the conic section which by (III′) is associated to A. As A traverses the fixed line a, the corresponding line p becomes tangent to a curve. Let Q be the point of contact between this curve and p. Finally let Q_0 be the point of intersection of p and q. The systems of ∞^3 curves with properties (I′), (II′) and (III′) are such that, when A is fixed but a varies, the correspondence between Q and Q_0 is, in general, a nonparabolic projectivity Π. One invariant point of Π is A and the other is a point V. (IV′) The invariant cross ratio (AV, QQ_0) of the projectivity Π has the value $j=2/3$.

Next the author introduces the point-to-line correspondence C associated to any system of ∞^3 curves with properties (I′)–(IV′). Let B and B^* be any two distinct fixed points in the plane and let A be a variable point. Construct the lines $a=(AB)$ and $a^*=(AB^*)$. By (I′) and (II′), there are two points P and P^* on a and a^*, respectively, and a single line p through A. From (I′), (II′) and (III′), it is found that there are two points Q and Q^* on p. Let Z be the intersection of the lines (B^*Q) and a, and Z^* the intersection of the lines (Q^*B) and a^*. The point of intersection λ of the lines (PP^*) and (ZZ^*) and the point A determine the line v. Let the line w through λ be such that the lines (PP^*), (ZZ^*), v and w form a constant cross ratio k. The correspondence C is the one from A to w. It may be designated by $C(B, B^*, k)$. The author studies those correspondences C for which the constant cross ratio $k=6$. Any such transformation is denoted by $C(6)$. All $C(6)$ of the polar type are determined, that is, those for which the induced projectivity between the lineal elements through A and those on the corresponding line w is an involution. (V′) The associated correspondence $C(6)$ must be polar. Kasner's property (I) and the author's (I′) serve to characterize any system of ∞^3 curves represented by a differential equation of the type (G). *J. De Cicco.*

Kasner, Edward and De Cicco, John. Synthetic solution of the inverse problem of dynamics. Proc. Nat. Acad. Sci. U. S. A. 28, 413–417 (1942). [MF 7287]

In general four plane trajectories through a point determine the direction of the force field at the point. The authors give a synthetic ruler and compass construction for the direction, as well as a geometric criterion for the exceptional indeterminate case. *P. Franklin.*

Levi, B. Definition and conditions for the existence of the tangent and the osculating circle at a point of a curve. Math. Notae 2, 11–34 (1942). (Spanish) [MF 6739]

This paper concerns itself with the condition on the function $y=f(x)$ in order that the curve C determined by

the function shall have a tangent line or osculating circle at a point O according to the following definitions: (1) Let A be a point on C. If OA approaches a limit as A approaches O, this limit is called the tangent to C at O. (2) Let A and B be points distinct from O. If AB approaches a limit as A and B approach O independently this limit is called the tangent to C at O. (3) Let C have a tangent at O, and let A be a point on C distinct from O. If the circle determined by the tangent and A approaches a limit as A approaches O, this limit is called the osculating circle at O. (4) Let A and B be two points distinct from O. If the circle OAB approaches a limit as A and B approach O independently this limit is called the osculating circle to C at O. (5) Let A, B, C be points of C distinct from O. If the circle ABC approaches a limit as A, B, C approach O independently, this limit is called the osculating circle to C at O. Examples are given illustrating the equivalence or nonequivalence of these definitions. *V. G. Grove* (East Lansing, Mich.).

Mirguet, Jean. Sur une classe de surfaces sans paratingent troisième. Bull. Sci. Math. (2) **64**, 257–268 (1940). [MF 6788]

A paratingent is of strictly even rank if it is of rank $2k$ without being of rank $2k+1$. The principal results of this paper are summarized in the following theorem: If a surface has at every point a finite number of paratingents of rank higher than the first, then at each point on the surface (1) the contingent is polyhedral, dihedral or plane and the paratingents of strictly even rank are tangent, and (2) there exist at most two paratingents of strictly even rank, except perhaps at a countable set of points.

C. C. Torrance (Cleveland, Ohio).

RELATIVITY

***Bergmann, Peter Gabriel. Introduction to the Theory of Relativity.** Prentice-Hall, Inc., New York, 1942. xvi+287 pp. $4.50.

In a foreword, Einstein describes this book as "primarily a textbook for students of physics and mathematics, which may be used either in the classroom or for individual study." It is, however, an unusual textbook; though it starts at the beginning of the subject, it carries the reader into the domain of recent researches. To accomplish this in so short a book is a remarkable achievement. The theory of the expanding universe is not covered, but that is the only omission of note. The book has three parts: (I) Special Theory; (II) General Theory; (III) Unified Field Theory. The development follows the semi-inductive plan preferred by physicists, rather than the axiomatic pattern of the mathematician. This does not mean, however, that any attempt is made to avoid essential mathematical formalism. This is given in considerable detail; two chapters are devoted to the theory of tensors in n-space.

The first three chapters of the book deal with prerelativistic mechanics and optics, and the difficulties arising therein which the theory of relativity was created to resolve. After these preliminaries, the special theory is covered fairly thoroughly: Lorentz transformation, mechanics of mass points, electrodynamics, continuous matter, applications. Most of the chapters in part (I) end with sets of problems.

Part (II) starts with an inductive approach to the general theory of relativity through the "principle of equivalence." Interesting as this is from a historical point of view, some of the arguments strike the reviewer as confusing. Thus, in discussing the definition of an inertial system as one sufficiently far removed from all other bodies, the author states [p. 156] that "sufficiently far" has no Lorentz-invariant significance. This is not true, for a space-time metric determines an invariant "distance" between two world-lines. A chapter is devoted to the field equations, with emphasis on nearly flat fields, the linear approximation and coordinate conditions. The next two chapters deal with rigorous solutions of the field equations (including the rotationally symmetric solutions of Weyl and Levi-Civita) and with the experimental tests of the general theory. The last chapter of Part (II) gives a fairly detailed account of the Einstein-Infeld-Hoffmann theory of the equations of motion. This chapter ends with the statement of a problem: to derive the equations of motion of electrically charged mass points. It might be noted that this problem has been solved by Infeld and Wallace [Phys. Rev. (2) **57**, 797–806 (1940); Amer. J. Math. **63**, 729–740 (1941); these Rev. **1**, 274; **3**, 212]. Part (III) is the shortest, containing only 35 pages. There brief accounts are given of the following: (1) Weyl's gauge-invariant geometry (following a formalism different from, but equivalent to, Weyl's); (2) Kaluza's five-dimensional theory and the projective field theories (Veblen, Hoffmann, Pauli); (3) a generalization of Kaluza's theory (Einstein, Mayer, Bargmann, G. Bergmann).

J. L. Synge (Toronto, Ont.).

Einstein, A. Demonstration of the non-existence of gravitational fields with a non-vanishing total mass free of singularities. Univ. Nac. Tucumán. Revista A. **2**, 5–15 (1941). (Text in Spanish and English) [MF 6744]

It is shown that a contradiction is obtained from the requirements: (1) g_{ij} and $g_{ij}+\delta g_{ij}$, where δg_{ij} is an infinitesimal of first order, be everywhere regular; (2) both satisfy $R_{ij}=0$ everywhere; (3) both satisfy the boundary condition that for large distances from the origin of coordinates $g_{ij}\to\eta_{ij}$, where η_{ij} is the metric tensor of Minkowski space. The asymptotic values of the solution of the equations $R_{ij}=0$ of the form $g_{ij}=\eta_{ij}+\gamma_{ij}$, where $\gamma_{st}=-(2m/r)\delta_{st}$, $s, t=1, 2, 3$, $\gamma_{44}=-2m/r$, δ_{st} is the Kronecker delta and m is a constant interpreted as the total mass, are used in evaluating the integral over a four-dimensional volume of $(-g)^{\frac{1}{2}}g^{ik}\delta R_{ik}$, which is shown to be equal to the divergence of a vector density. It is then concluded: "This contradiction can only arise from the inexactitude of the hypothesis that there exists a solution free of singularities belonging to a total mass different from zero." *A. H. Taub.*

Band, William. Comparison spaces in general relativity. Phys. Rev. (2) **61**, 702–707 (1942). [MF 6742]

The author considers a Riemannian metric $ds^2=g_{\mu\nu}dx^\mu dx^\nu$ and, in addition, an arbitrary comparison metric

$$d\sigma^2=\gamma_{\mu\nu}dx^\mu dx^\nu.$$

For arbitrary choice of the $\gamma_{\mu\nu}$ Rosen's formalism [Phys. Rev. (2) **57**, 147–155 (1940); these Rev. **1**, 183] carries through. When $d\sigma$ is the metric of a flat space, that is, tangent to the Riemannian space, it is shown that the physical interpretation of the apparent force depends on the point of tangency of the two spaces. When the comparison is a de Sitter universe, it is shown that differences

between the two spaces correspond to a continuous distribution of matter. *M. Wyman* (Ottawa, Ont.).

Band, William. Is space-time flat? Phys. Rev. (2) **61**, 698–701 (1942). [MF 6741]

Although the author does not answer the question asked in the title of this paper he points out, by example, some of the difficulties that one must overcome in adopting a flat space-time point of view. A commentary is also given on an article by N. Rosen [Phys. Rev. (2) **57**, 147–155 (1940); these Rev. **1**, 183]. *M. Wyman* (Ottawa, Ont.).

Band, William. Vector analysis in special relativity. Amer. J. Phys. **8**, 162–164 (1940). [MF 6508]

Racine, C. Contribution to the relativistic problem of n bodies. I. J. Indian Math. Soc. (N.S.) **5**, 156–164 (1941). [MF 6978]

Racine, C. Contribution to the relativistic problem of n bodies. II. J. Indian Math. Soc. (N.S.) **5**, 165–178 (1941). [MF 6979]

The author claims to have confirmed Levi-Civita's result that the center of gravity of two particles has an acceleration, according to the general relativity theory. The method of calculation is somewhat similar to that of Levi-Civita and the result is the same. [It may be useful to straighten the record of this problem. After Levi-Civita published his result [Amer. J. Math. **59**, 9–22 (1937)], the relativistic equations of motion for two bodies were derived by a different method by Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)]. H. P. Robertson deduced from these equations that the center of gravity of two particles does not accelerate [Ann. of Math. (2) **39**, 101–104 (1938)]. This result was obtained independently by Eddington and Clark [Proc. Roy. Soc. London. Ser. A. **166**, 465–475 (1938)] and again by a different method by Fock [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 25–27 (1941); these Rev. **3**, 212]. [The last two papers are not quoted in the work reviewed.] Finally H. P. Robertson removed the discrepancy by discovering the mistake in Levi-Civita's calculations and showing that its removal leads to a uniform motion of the center of gravity. This result was not published and I owe its knowledge to a private communication.] *L. Infeld* (Toronto, Ont.).

Lanczos, Cornelius. Matter waves and electricity. Phys. Rev. (2) **61**, 713–720 (1942). [MF 6743]

This paper is a summary (in which mathematical details are omitted) of a unified field theory based on the action principle

$$(*) \qquad \delta \int S_{\alpha\beta} S^{\alpha\beta} d\tau = 0,$$

where $S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} R$, $R_{\alpha\beta}$ is the Ricci tensor and R is the scalar curvature. The tensor $S_{\alpha\beta}$ is referred to as the "matter" tensor and the equations $S_{\alpha\beta} = 0$ characterize the "basic metric plateau" (space with no matter). It is stated that these equations become, as a consequence of the field equations, $R_{\alpha\beta} = \lambda g_{\alpha\beta}$, where λ is a constant of integration. Reasons are given for assuming λ large. The variations of (*) are carried out under the subsidiary condition that the divergence of $S^{\alpha\beta}$ vanishes. This introduces a vector function φ_α which is interpreted as the vector potential. The author states in the introduction that the electromagnetic fields are determined in terms of this vector by the equations

$$(\dagger) \qquad F_{\alpha\beta} = \frac{\partial \varphi_\alpha}{\partial x^\beta} + \frac{\partial \varphi_\beta}{\partial x^\alpha}$$

instead of the usual ones

$$F_{\alpha\beta} = \frac{\partial \varphi_\alpha}{\partial x^\beta} - \frac{\partial \varphi_\beta}{\partial x^\alpha}.$$

Since the author's equations (4.3) relating the matter tensor $S_{\alpha\beta}$ with the vector potential contain the covariant form of the right member of (†), it would seem that the electromagnetic field tensor and the matter tensor are the same. This theory would then differ markedly from the usual one when the matter tensor is a quadratic function of the field strength. Moreover, this theory, if based on (†), would not be gauge invariant and hence would differ in another respect from the usual theory. *A. H. Taub* (Princeton, N. J.).

Stueckelberg, E. C. G. Remarque à propos de la création de paires de particules en théorie de relativité. Helvetica Phys. Acta **14**, 588–594 (1941).

Stueckelberg, E. C. G. La mécanique du point matériel en théorie de relativité et en théorie des quanta. Helvetica Phys. Acta **15**, 23–37 (1942).

(i) Classical theory. According to the theory of relativity, a particle of mass m and charge e when subjected to external gravitational and electromagnetic forces describes a world line $x^\mu = q^\mu(s)$ determined by

$$(A) \qquad m dw^\mu/ds = -m\Gamma^\mu_{\alpha\beta} w^\alpha w^\beta + eB^{\mu\nu} w_\nu; \quad w^\mu = dq^\mu/ds.$$

s is the proper time ($ds^2 = g_{\mu\nu} dq^\mu dq^\nu$), $B_{\mu\nu} = -B_{\nu\mu}$ are the components of the electromagnetic field, and $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols. The author introduces a parameter λ (instead of s). Denoting the derivatives $dq^\mu/d\lambda$ by $\dot{q}^\mu$, he obtains the modified equations of motion

$$(B) \qquad d\dot{q}^\mu/d\lambda = -\Gamma^\mu_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta + eB^{\mu\nu} \dot{q}_\nu.$$

The expression $m^2 = -\dot{q}_\mu \dot{q}^\mu$ is a constant of integration of the system (B) and defines the mass of the particle. To the two cases $\dot{q}^4 > 0$ and $\dot{q}^4 < 0$ correspond particles with charges e and $-e$. Together with Einstein's gravitational equations and Maxwell's equations, the equations (B) are derived from a variational principle. By standard methods the author finds the expressions for the energy-momentum tensor and the four vector current density, which both satisfy conservation laws. The equations of motion (B) are derived from $\delta \int L d\lambda = 0$, with the Lagrangian

$$L = \tfrac{1}{2} \dot{q}_\mu \dot{q}^\mu + e\phi_\mu(q) \dot{q}^\mu, \quad B_{\mu\nu} = \frac{\partial \phi_\nu}{\partial x^\mu} - \frac{\partial \phi_\mu}{\partial x^\nu}.$$

Momenta p_μ are defined by $\partial L/\partial \dot{q}^\mu = \dot{q}_\mu + e\phi_\mu$, and a canonical system of equations is set up in close analogy with Hamiltonian mechanics, the parameter λ replacing the time. Similarly, a Hamiltonian is introduced by $R(p, q) = -L + p_\mu \dot{q}^\mu$. Without destroying the relativistic invariance of the system (B), one may add an extra term $K^\mu(q)$ to its right side, so that the expression m^2 will, in general, no longer be a constant of integration. In particular, there might exist a world line C (for a suitable field K^μ) which cuts every hyperplane $t = \text{const.}$ in two points for $t < t_0$, and no point of which has a t-component $t > t_0$, such that for a certain value of the parameter λ the hyperplane $t = t_0$ is tangent to C. Then C would represent two particles which exist for $t < t_0$ and are annihilated at the time t_0. (In a similar way one might represent the creation of a pair of particles by interchanging past and future.) As the author points out,

a theory of this type must be rejected because it violates the principle of causality.

(ii) Quantum theory. Gravitation is neglected, and no extra field K^μ is assumed. Starting with the Hamiltonian form of the equations of motion, the author, in close analogy wih Schrödinger's theory, introduces a complex function $\psi(q^1, q^2, q^3, q^4, \lambda)$ which satisfies the wave equation $ih\partial\psi/\partial\lambda = R(p, q)\psi$ and is normalized by the condition $\int|\psi|^2 dq^1 dq^2 dq^3 dq^4 = 1$. The p_μ denote the differential operators $-ih\partial/\partial q^\mu$. Expectation values of Hermitian operators are defined as integrals extended over all four coordinates; they depend on λ. By integrating them over λ one obtains constant mean values. The energy turns out to be positive; the electric charge density may have both signs. The creation and annihilation of pairs of particles is a consequence of the theory, the main reason being the following: It is clear that a ψ-wave is scattered, that is, refracted by an electric field. Because of the wave equation, with every refracted wave is connected a reflected one, which, as the analysis shows, represents a particle of opposite charge. It is, therefore, unnecessary to introduce an extra field K^μ into the wave mechanical theory. *V. Bargmann.*

MECHANICS

Draper, Glen H. Newtonian mass, its definition and determination. Philos. Mag. (7) **33**, 476–478 (1942). [MF 7074]

Badell, E. and Maseda, M. A. Components of acceleration in curvilinear coordinates. Revista Soc. Cubana Ci. Fís. Mat. **1**, 2–3 (1942). (Spanish) [MF 7190]

Astronomy

García, Godofredo. Generalization of Sundman's fundamental equality to the case of more than three bodies. Proc. Nat. Acad. Sci. U. S. A. **28**, 425–427 (1942). [MF 7290]

García, Godofredo. Generalization of the inequality of Sundman to the case of more than three bodies and to the case of a gravitational gas. Proc. Nat. Acad. Sci. U. S. A. **28**, 428–432 (1942). [MF 7291]

Sundman's inequality [Acta Soc. Sci. Fennicae **35**, no. 9 (1909); Acta Math. **36**, 105–179 (1913), in particular, p. 149] for the function $R^2 = \sum m_\nu r_\nu^2$, where r_ν is the distance of the νth body from the center of gravity, is here generalized from $n = 3$ to arbitrary n and then, by passage to the limit, to a continuous medium. The result for arbitrary n has been known since 1909. For a full discussion, see Wintner's fundamental treatise [The Analytical Foundations of Celestial Mechanics, Princeton Mathematical Series, vol. 5, Princeton University Press, Princeton, N. J., 1941, pp. 252, 428; these Rev. **3**, 215]. *W. Kaplan.*

García, Godofredo. The three-body problem. Revista Ci., Lima **44**, 159–186 (1942). (Spanish) [MF 7114]

Using purely vector methods, the author derives several classical results of Lagrange and Sundman. [These results are included in G. D. Birkhoff, Dynamical Systems, Amer. Math. Soc. Colloquium Publ., vol. 9, American Mathematical Society, New York, 1927, ch. 9.] *W. Kaplan.*

Pedersen, Peder. Die Librationsellipsen um die Dreieckslibrationspunkte im allgemeinen Dreikörperproblem. Danske Vid. Selsk. Math.-Fys. Medd. **19**, no. 7, 25 pp. (1941). [MF 6799]

All the results of this paper are given in a paper by the reviewer [Amer. J. Math. **50**, 613–629 (1928)]. [Some additional results are given by the reviewer in Amer. J. Math. **51**, 595–598 (1929).] *H. E. Buchanan.*

Godart, Odon. Mouvement de particules chargées dans le champ d'un dipôle magnétique sur une famille d'orbites périodiques. Univ. Nac. Tucumán. Revista A. **2**, 109–129 (1941). (French. Spanish summary) [MF 6752]

The author sets up the differential equations for the movement of a charged particle in the field of a magnetic dipole. These equations are not integrable by the usual methods of analysis. He derives two integrals which resemble the energy integral and the integral of areas in the two body problem. He finds a particular solution and discusses the analytic continuation of this solution as was done by Poincaré and Moulton. In particular, he studies an important family of periodic orbits which play an important role in the theory of cosmic rays. His results are obtained by a combination of numerical calculations and mechanical integration aided by the geometrical aspects of the problem. The paper would have been much more satisfactory to the reviewer if the author had shown more in detail how he determined his characteristic exponents and how he made use of his periodicity equations. *H. E. Buchanan.*

***Chandrasekhar, S. Principles of Stellar Dynamics.** University of Chicago Press, Chicago, Ill., 1942. x+251 pp. $5.00.

During the last fifty years the theory of the distribution of the stars and of their velocities has made extensive progress, due both to an improvement in observational technique, with the resulting accumulation of a large amount of evidence, and to the brilliant theoretical contributions of a number of men, including Jeans, Oort, Lindblad and Chandrasekhar. In this book these theoretical researches are described with emphasis on two points of view, that of statistical mechanics and that of classical mechanics. The first point of view is based on the fact that stars are numerous and suggest the molecules of a gas, the second on the facts that the stars are far apart, that collisions are negligible in frequency, and hence that there is a dynamical problem equivalent to the n-body problem or to that of a particle moving in a nonsteady force field.

The first chapter, "Kinematics," postulates the existence of a local standard of rest, which is equivalent to the assumption that, at least locally, the stars can be regarded as imbedded in a fluid motion. Correspondingly a fundamental distribution function Ψ is defined which gives the density of distribution of stars of each velocity at each point. It is shown that near the sun the variation of velocity distribution with direction is described empirically by Schwarzschild's law.

The second chapter, "The Time of Relaxation of a Stellar System," is devoted to a determination of how negligible collisions are. On the basis of precise analysis of the two-

body problem, formulas for energy change and angular deflection at collision are derived. By use of empirical data on star distribution an estimate is then made for the probable time it would take for collisions to affect seriously the star's orbit. This is the time of relaxation and, as in the theory of fluids, a corresponding mean free path and a notion of stellar viscosity can be defined, though the numerical values make them almost meaningless in the usual sense.

Chapter III, on "Galactic Dynamics," now takes advantage of the established negligible frequency of collisions in our own galaxy to consider all stars as moving under the influence of a common nonsteady external force field, the motion being described by canonical equations with the same H for all stars. Liouville's theorem is then applied to the density Ψ of chapter I, so that Ψ appears as an integral of the equations of motion. The fundamental question then asked is the following: what form can the potential have if the equations admit of an integral of the form prescribed by Schwarzschild's law? This implies existence of an integral which is a quadratic form in the velocities. The question is answered by a partial differential equation which can be decomposed into twenty separate partial differential equations. Special assumptions on the potential are then made, and corresponding conclusions obtained from the partial differential equations. For example, if the potential is steady, the potential has axial symmetry.

Chapter IV, "General Dynamics of Stellar Systems: Spiral Structure," makes the assumption that the local velocity distribution Ψ is spherical and further conclusions are obtained about the potential. The stability of circular orbits in a potential field with both an axis and a plane of symmetry is considered. The conclusions are applied to the field of a homogeneous oblate spheroid and then (following Lindblad) to an explanation of the spiral structure of nebulae.

The fifth chapter, "The Dynamics of Star Clusters," treats a star cluster as a case of the n-body problem, with n very large, so that statistical reasoning can be used. A famous identity of Lagrange and Jacobi is derived and applied to determining the average velocity, the time of relaxation and the mean free path. In contrast to the third chapter, collisions now turn out to be of significance. The rate of escape of stars from the cluster is analyzed. Next the problem of a cluster whose center of gravity is moving in a circle (under influence of a large external force) is considered: a smoothed out distribution for the stars in the cluster is assumed, a stability criterion is derived and the effect of collisions is discussed.

The book is written with exceptional clarity, and a number of plates and tables enhance its readability. In general the constant comparison of theory with experiment adds to the significance of the results. At the end of each chapter there is a bibliography with a useful discussion of the historical evolution of the ideas. The book should prove stimulating to astronomer, mathematician and physicist.

W. Kaplan (Ann Arbor, Mich.).

Chandrasekhar, S. and Krogdahl, Wasley. A note on the perturbation theory for distorted stellar configurations. Astrophys. J. 96, 151–154 (1942). [MF 7018]

Coutrez, R. L'équilibre dynamique des systèmes stellaires. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 295–308 (1941). [MF 6890]

A stellar system is considered as composed of n subsystems of stars, each subsystem characterized by the mass $m \pm \delta m/2$ of its members. In addition a continuous interstellar material is present. Coordinates fixed at the center of gravity of the whole stellar system are introduced, and it is shown that only the forces derivable from the gravitational potential are of importance for study of the dynamic equilibrium. Because of the long time of relaxation, the effects of encounters are also neglected. The equations of motion of a member of a system are written down, and assumed to possess the integrals of energy and angular momentum. Symmetry about the plane $Z=0$ is also assumed.

A distribution function in configuration space must have the form $f=f((H-\mu')/\sigma^2)$, where H is the energy, μ' the "chemical potential" per particle (star), and σ a measure of the dispersion in velocities. An exponential (Boltzmannian) function is chosen, and μ' represented by a power series in the angular momentum. By breaking this power series off after the quadratic term, a spheroidal distribution function is obtained as a special case. This case is discussed. With the quadratic term in μ' disappearing, a random distribution arises, and no differential rotation can exist. The possibility of ring formed systems are shown.

G. Randers (London).

Krogdahl, Wasley. The equilibrium of a perfect compressible fluid configuration. Astrophys. J. 96, 124–150 (1942). [MF 7017]

The usual perturbation method is applied to the study of the equilibrium of an autobarotropic fluid, that is, one in which a relation $P=P(\rho)$ between pressure and density holds. The calculations follow exactly those carried through earlier for the more special case of a polytropic fluid ($P=k\cdot\rho^n$) by other investigators. First the case of slow rotation is considered, and the calculations carried through to the second order in the quantity $\epsilon=\omega^2/4\pi G\rho_c$, where ω is the angular velocity, ρ_c the central density. Then the tidal perturbation of a point mass on the configuration is calculated to the same order. Finally, the two effects are considered as acting simultaneously. This calculation is not carried through and only trivial results are obtained. The theory of the two first sections are applied to three special models studied recently by Henrich.

G. Randers (London).

---

53 *Hydrodynamics, Aerodynamics, Acoustics*

Kravtchenko, Julien. Sur le problème de représentation conforme de Helmholtz; théorie des sillages et des proues. J. Math. Pures Appl. (9) 20, 235–303 (1941). (conclusion) [MF 6835]

As in the previous chapters [J. Math. Pures Appl. (9) 20, 35–234 (1941); these Rev. 3, 219], the method employed to solve the uniqueness problem is that of J. Leray [Comment. Math. Helv. 8, 149–180, 250–263 (1936)]. The chief concern here is to discover those cases having unique solution. Two infinitesimally close solutions are compared with the aid of general functional theory. The treatment is facilitated by an idea due to A. Weinstein and a lemma due to K. Friedrichs. It is shown that, for flow between two infinitely long parallel walls, the wake problem has a unique solution for (i) bodies symmetrical with respect to an axis parallel to the wall placed symmetrically with respect to the channel, (ii) symmetrical and unsymmetrical

bodies convex to the flow, including the case of flow limited by one wall only. For the bow problem, the solution is unique if (i) the body is symmetrical and pointed (in the sense of Leray) and is placed symmetrically with respect to the channel, (ii) the body is represented by a circular arc and the complete circle obtained by extending the circular arc is in the interior of the flow domain. These results are the generalization of those obtained by Leray.

H. S. Tsien (Pasadena, Calif.).

Kolscher, M. Unstetige Strömungen mit endlichem Totwasser. Luftfahrtforschung **17**, 154–160 (1940). [MF 6559]

The author discusses wakes of finite extension behind obstacles and shows that such wakes are possible from the mathematical and kinematical viewpoints. However the author does not point out that finite wakes are impossible from the dynamical viewpoint [see M. Brillouin, Ann. Chim. Phys. Ser. A. **23**, 145–230 (1911), p. 170 in particular]. The last part of the paper is devoted to an extension of a paper by C. Schmieden [Ing.-Arch. **3**, 356–370 (1932); ibid. **5**, 373–375 (1934)]. It is therefore open to the same criticism as has been made of Schmieden's paper by the reviewer [A. Weinstein, Enseignement Math. **35**, 107–125 (1936), in particular, p. 123] and J. Leray [Comment. Math. Helv. **8**, 149–180, 250–263 (1935), in particular, p. 152]. See also J. Kravtchenko [see the preceding review].

A. Weinstein (Toronto, Ont.).

***von Kármán, Theodore. Problems of flow in compressible fluids.** University of Pennsylvania Bicentennial Conference, Fluid Mechanics and Statistical Methods in Engineering, pp. 15–39 (1 plate). University of Pennsylvania Press, Philadelphia, Pa., 1941.

This paper opens with a discussion of the various fields of hydraulics and hydrodynamics in which consideration of the density variations is essential. Gas dynamics "in hydraulics fashion" is then discussed, as exemplified by the one-dimensional theory of flow through a nozzle of variable cross section. This theory is given in detail. Shock waves are also discussed, and certain limitations of present theories of such waves are pointed out. The analogy between one- and two-dimensional flows of compressible fluids and flows of incompressible fluids in open channels is explained.

In a section devoted to problems of subsonic flow in aeronautics the author first sets up the nonlinear partial differential equation for the stream function. The approximate solution suggested by Glauert and Prandtl, which is based on the assumption that the flow consists of a small perturbation of a parallel flow, is shown. The method employing the hodograph transformation is also described; by use of this transformation the partial differential equation is made linear. von Kármán and Tsien have employed this transformation together with the linear approximation to the isentropic equation of state used by Chaplygin, Demtchenko, A. Busemann and others. The results of Tsien's calculations are discussed.

The paper closes with a descriptive account of the application of the theory of compressible fluids to exterior ballistics. These applications usually involve axially symmetric solutions of the nonlinear differential equation. Recent investigations in this field, in the supersonic regime, are described.

W. R. Sears (Inglewood, Calif.).

Stewart, H. J. The energy equation for a viscous compressible fluid. Proc. Nat. Acad. Sci. U. S. A. **28**, 161–164 (1942). [MF 7036]

The correct form of the energy equation for the flow of a compressible viscous fluid is derived in the form

$$\rho\frac{DQ}{Dt}=\rho\frac{D}{Dt}\{I+V+\tfrac{1}{2}u_iu_i\}+\frac{\partial}{\partial x_i}(pu_i)-\frac{\partial}{\partial x_j}(\tau_{ij}u_i),$$

where Q is the heat added per unit mass of the fluid, I is the internal energy and V the potential of the external forces. The author shows that the equation

$$\rho\frac{DQ}{Dt}=\rho\frac{D}{Dt}\{I+V+\tfrac{1}{2}u_iu_i\}+\frac{\partial}{\partial x_i}(pu_i)-u_i\frac{\partial}{\partial x_j}\tau_{ij}$$

given in some textbooks on dynamic meteorology is incorrect due to the omission of the terms $\tau_{ij}\partial u_i/\partial x_j$ which represent the kinetic energy dissipated per unit time and volume by the action of the viscous stresses.

H. W. Liepmann (Pasadena, Calif.).

Mohr, Ernst. The Navier-Stokes stress principle for viscous fluids. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1029**, 11 pp. (1942). (2 plates) [MF 7292]

Translation of an article in Luftfahrtforschung **18**, 327–330 (1941); these Rev. **3**, 285.

Keulegan, Garbis H. Equation of motion for the steady mean flow of water in open channels. J. Research Nat. Bur. Standards **29**, 97–111 (1942). [MF 6964]

In the calculation of the steady varied flow of liquids in a channel, the problem can be made to be one-dimensional by introducing an average flow velocity over each cross section of the channel and a velocity-distribution factor. Coriolis investigated the change in the forms of energy along the channel and introduced the "mean cube" coefficient obtained by averaging the cube of the local velocities over the cross section of the channel. Boussinesq used the momentum equation and introduced the "mean square" coefficient. There is disagreement in the literature regarding the use of these two coefficients. Bakhmeteff [Theodore von Kármán Anniversary Volume, California Institute of Technology, Pasadena, Calif., 1941, pp. 59–65; these Rev. **3**, 94] claimed that the "mean cube" coefficient should not be used whenever the problem is to compare energy conditions at different stations, such as the varied flow equation.

By starting from the exact Euler's equations involving Reynolds stresses, the present author shows that both coefficients can be used provided proper interpretations are given to the resistance coefficient. Using the energy equation, the resultant flow equation contains the "mean cube" coefficient and a resistance coefficient related to the rate of energy loss. Using the momentum equation, the resultant equation contains the "mean square" coefficient and a resistance coefficient related to the wall friction. It is pointed out that this difference in meaning of resistance coefficients is usually not observed in using Manning's "n" in the flow equation for open channels. [It seems to the reviewer that the term involving N_1 in eq. (34) is of higher order and thus should be dropped. However, this does not affect the conclusion.]

H. S. Tsien (Pasadena, Calif.).

Seth, G. R. and Gupta, Qabul Chand. Method of images applied to waves in canals. Proc. Benares Math. Soc. (N.S.) 2, 25–31 (1940). [MF 6645]

Havelock, T. H. The drifting force on a ship among waves. Philos. Mag. (7) 33, 467–475 (1942). [MF 7073]

It is found that ocean waves increase the average steady resistance of the ship. The reflection or scattering of these waves by the surface of the ship will cause some increase in resistance, but as shown by the author in a previous paper [T. H. Havelock, Proc. Roy. Soc. London. Ser. A. 175, 409–421 (1940)], this effect is very small. In this paper the author made some tentative calculation for the increase in resistance of a ship due to heaving and pitching in a train of waves. Because of the presence of damping in such motions, there are phase lags between the motions of the ship and the waves. It is shown that these phase lags give an average steady drifting force. For a rough estimation, the waves are assumed to be of small height and irrotational, and the disturbance effect due to the presence of the ship is neglected. The result gives a drifting force of same order of magnitude as that of experiments.

H. S. Tsien (Pasadena, Calif.).

Havelock, T. H. The damping of the heaving and pitching motion of a ship. Philos. Mag. (7) 33, 666–673 (1942). [MF 7229]

When a body is floating freely in water and subjected to a periodic force $E\cos pt$, the water pressure on the surface of the body is composed of three parts. The first part is the hydrostatic buoyancy proportional to the displacement ζ. Then there is a part proportional to $\ddot{\zeta}$ which can be considered as an additional inertia term. In this paper, the author treats the third part which is proportional to $\dot{\zeta}$, say $N\dot{\zeta}$. Instead of calculating the fluid pressure, the mean rate of propagation of energy outward in the wave motion is determined. Equating this to the mean value of $N\dot{\zeta}^2$, the author obtains N for the frequency p. Now if the periodic external force is absent, and the rate of damping of the resultant motion is small, $N\dot{\zeta}$ can be considered as damping force of the motion and used to calculate the logarithmic decrement δ of the amplitude of the motion.

To carry out such calculations for the heaving motion the author used a distribution of two-dimensional point source immersed at a constant depth equal approximately to the draught. For a typical example, the value of δ is 1.4. This check will vary with the reported experimental data. However, the motion can no longer be considered as slowly damped. Therefore, the result of calculation is only qualitative. A distribution of three dimensional point source is used for the pitching motion and similar numerical results are obtained. *H. S. Tsien* (Pasadena, Calif.).

Høiland, Einar. On the stability of the circular vortex. Avh. Norske Vid. Akad. Oslo. I. 1941, no. 11, 24 pp. (1941). [MF 6800]

The author considers a field of angular velocity $\Omega(R, H)$, where H and R denote coordinates parallel and normal to the axis of rotation, respectively. An integration of the Eulerian equations of motion along a closed path eliminates the pressure p and furnishes the equilibrium conditions for the vortex in integral form. If the vector $\bar{\rho}$ denotes a small meridional displacement, the density q and the radial acceleration $\bar{a}$ can be written $q=q_0+\Delta q$, $\bar{a}=\bar{a}_0+\Delta\bar{a}$; if furthermore $\partial^2\bar{\rho}/\partial t^2=-\sigma^2\bar{\rho}$, the following equation for the frequency σ of the perturbation is obtained

$$\sigma^2=-\frac{\int \Delta q(\bar{g}+\Omega^2\bar{R})\cdot\delta\bar{r}-\int q\Delta\bar{a}\cdot\delta\bar{r}}{\int q\bar{\rho}\cdot\delta\bar{r}}, \tag{1}$$

where $\bar{g}$ is a conservative force, that is, the gravity. The sign of the nominator in (1) determines the stability. The author discusses the stability of the general baroclinic (that is, $\nabla q\times\nabla p\neq 0$) vortex and the special cases of the barotropic ($\nabla q\times\nabla p=0$) and potentiotropic ($\nabla q\times\bar{g}=0$) vortex. It is shown that (1) always leads to two stability conditions. In the case of the barotropic and the incompressible potentiotropic vortex these conditions can be easily divided in a static and a dynamic stability condition. In the case of the general baroclinic vortex this division is not so simple. The discussion shows that the stabilizing and destabilizing effects in the baroclinic case are effects of a quite new type.

H. W. Liepmann (Pasadena, Calif.).

Van Mieghem, Jacques. Sur la vitesse de propagation des cyclones. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 309–317 (1941). [MF 6891]

In a previous paper [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. 19, fasc. 3, 65 pp. (1941); these Rev. 3, 219] the author derived new approximate expressions for the horizontal components of the wind. These equations are used to show that the velocity of propagation c of a cyclone is always smaller than the velocity U of the fundamental current. The height H of a "homogeneous atmosphere" is defined by $P_0=gQH$, where P_0 is the pressure on the ground, Q the (constant) density and g the constant of gravity. Similarly a small perturbation p_0^* of the pressure defines a deviation in height $p_0^*=gQ\zeta$. Assuming ζ of the form $\zeta=Z(y)\sin(\alpha x-ct)$, the following perturbation equation is obtained

$$Z''-\left\{\alpha^2-\frac{1}{U-c}\left(\frac{\lambda^2 C}{gH}+\frac{\partial\lambda}{\partial y}-\frac{\partial^2 U}{\partial y^2}\right)\right\}Z=0,\quad \lambda=2\omega\sin\varphi.$$

ω denotes the angular velocity of the earth, φ the latitude. By solving for c and discussing the terms of the equation the author shows that $c<U$. This result is in agreement with meteorological observation. *H. W. Liepmann.*

Schwez, M. Die Bestimmung der Vertikalkomponenten der Geschwindigkeit einer sich bewegenden Luftmasse mittels der hydrodynamischen Bewegungsgleichungen. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 1941, 467–473 (1941). (Russian. German summary) [MF 7327]

In der vorliegenden Arbeit wird mit Hilfe der Gleichungen der Hydromechanik eine Formel abgeleitet, die die Vertikalgeschwindigkeiten in den Schichten der Höhe von ca 4–5 Km aus dem Temperatur- und Druckbodenfeld zu berechnen gestattet. *Author's summary.*

Stockmann, W. Horizontale Unruhe der Meeresströmungen, aufgefasst als Turbulenzerscheinung grossen Stils. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 1941, 475–486 (1941). (Russian. German summary) [MF 7328]

Pekeris, Chaim L. Comments on T. E. W. Schumann's paper, "An investigation concerning G. I. Taylor's correlation coefficient of turbulence." Philos. Mag. (7) 33, 541–543 (1942). [MF 7091]

In a recent paper T. E. W. Schumann [Philos. Mag. (7) 32, 471–482 (1941); these Rev. 3, 221] objected to G. I. Taylor's proof that the expansion for the correlation coefficient R_ξ of a fluctuating quantity $p(t)$ contains only even powers of ξ. He has shown that in a certain case $R_\xi=e^{-2\xi^2/a^2}$. The present author first demonstrates how this expression for R_ξ can be deduced from the differential equation of the probability $F(p)$ without an explicit solution of the latter. He proceeds to show that the physical reason why the exponential correlation (which should be written $R_\xi=e^{-2\xi|\xi|/a^2}$) does not possess a Taylor series development at $\xi=0$ is that $\overline{p^2}$ is infinite. Functions that have an exponential correlation coefficient have no derivative and no higher derivatives. The author discusses these (Markoffian) processes, and also natural processes having ξ^2 and ξ^4 terms in the expansions of their correlation coefficients, the remainders being expressible only by Fourier integrals. He objects to the implication in Schumann's paper that, after periodic terms are eliminated, all natural phenomena are Markoffian.

W. R. Sears (Inglewood, Calif.).

Bolt, R. H., Feshbach, H. and Clogston, A. M. Perturbation of sound waves in irregular rooms. J. Acoust. Soc. Amer. 14, 65–73 (1942). [MF 7079]

Rosenblatt, Alfredo. On the theorem of Kutta-Joukowsky in aerodynamics. Actas Acad. Ci. Lima 5, 33–43 (1942). (Spanish) [MF 7112]

For the case of two-dimensional flow about a profile, the author deduces the known formulas for the force and moment in terms of the residues of w^2 and w^2z, respectively, where w is the "complex velocity" (the derivative of the complex potential) in the z plane. He applies these to the particular case of a thin-circular-arc profile by means of a conformal transformation. The results are then evaluated for (1) a flat plate, the circulation being determined by the usual condition of finite velocity at the trailing edge, (2) the case of zero circulation, (3) the case of no translatory flow (pure circulation). In the last case a discrepancy between the present paper and one of Cisotti [Atti Accad. Naz. Lincei. Rend. (6) 11, 122–128 (1930)] is discussed, and is attributed to an error in Cisotti's theory. There are a considerable number of typographical errors in the mathematical formulas. *W. R. Sears* (Inglewood, Calif.).

Behrbohm, Hermann. Über die Mindestabfanghöhe eines Flugzeuges. Luftfahrtforschung 19, 98–101 (1942). [MF 7434]

The author discusses various methods of determining the minimum altitude necessary to pull an airplane out of a dive. If the polar diagram of the plane is supposed to be a parabola, the problem leads to a Riccati equation which cannot be integrated in closed form. In order to obtain a more workable theory the author assumes that the difference between the propeller thrust and the drag remains constant during the pulling out. *W. Prager.*

Theory of Elasticity

Prager, W. Fundamental theorems of a new mathematical theory of plasticity. Duke Math. J. 9, 228–233 (1942). [MF 6350]

In a previous paper [Proc. 5th Intern. Congr. Applied Mech., Cambridge, Mass., 1938, pp. 234–237] the author has proposed stress strain relations, valid in elastic as well as in plastic regions, which in tensor notation and for an incompressible material are of the following form

$$\text{(a)}\quad \dot{s}_{ik}=2G\dot{e}_{ik}, \qquad \text{if } \dot{W}=\sigma_{pq}\dot{e}_{pq}<0,$$
$$\text{(b)}\quad \dot{s}_{ik}=2G[\dot{e}_{ik}-(\dot{W}/2\rho^2)s_{ik}], \qquad \text{if } \dot{W}>0.$$

In (a) and (b) the $s_{ik}=\sigma_{ik}-\sigma\delta_{ik}$ are the components of the stress deviator ($\sigma=\frac{1}{3}\sigma_{pp}$), G is the modulus of rigidity, ρ the yield stress in pure shear and $\dot{W}$ the rate of work done per unit volume. (Dots indicate differentiation with respect to time.) For an incompressible material for which $\dot{W}=\sigma_{pq}\dot{e}_{pq}=s_{pq}\dot{e}_{pq}$, $\dot{W}$ can be eliminated from (b) giving

$$\text{(b')}\quad 2G\dot{e}_{ik}=\dot{s}_{ik}+(s_{pq}\dot{s}_{pq}/(2\rho^2-s_{pq}s_{pq}))s_{ik}, \qquad \text{if } \dot{W}>0.$$

For bodies consisting of a material with these stress strain relations, the following boundary problems are formulated. (1) Assuming a given equilibrium state of stress σ_{ik}, determine its rate of change $\dot{\sigma}_{ik}$ caused by a given rate of change of boundary forces $\dot{f}_i$. (2) Again assuming a system of given σ_{ik}, determine the strain velocities $\dot{e}_{ik}$ for given boundary displacement velocities $\dot{u}_i$. For these two boundary problems (subject to the restriction that $\dot{W}>0$) the following variational principles, which are generalizations of the classical principles in the theory of elasticity, are proved. Theorem 1. "Among all systems $\dot{\sigma}_{ik}$ satisfying the condition of equilibrium $\dot{\sigma}_{ip,p}=0$ agreeing with the given rate of change $\dot{f}_i$ of the forces acting on the surface, the system actually set up minimizes the integral

$$U=\int (4G)^{-1}[\dot{s}_{pq}\dot{s}_{pq}+(s_{pq}\dot{s}_{pq})^2/(2\rho^2-s_{pq}s_{pq})]d\tau$$

extended over the total volume of the body." Theorem 2. "Among all systems of velocities $\dot{u}_i$ satisfying the condition of incompressibility $\dot{u}_{p,p}=0$ and assuming the given values on the surface, the system actually set up minimizes the integral

$$V=\int 2G[\dot{e}_{pq}\dot{e}_{pq}-(2\rho^2)^{-1}(\sigma_{pq}\dot{e}_{pq})^2]d\tau$$

extended over the total volume of the body."

E. Reissner (Cambridge, Mass.).

Stanton, Henry E. A mathematical theory of protoplasmic protrusions. I. Bull. Math. Biophys. 4, 93–100 (1942). [MF 7107]

Rashevsky [Advances and Applications of Mathematical Biology, University of Chicago Press, Chicago, 1940; these Rev. 2, 138] has applied the plastic-flow analogue of Betti's equations in deriving approximate equations of cellular elongation. In so doing he has regarded the cell as a cylinder with hemispherical caps. In this paper the author applies a similar procedure to determine the rate of and condition for elongation of a cylindrical protrusion from a spherical cell. The forces considered are those due to diffusing metabolites and to surface tension. *A. S. Householder.*

Kucharski, W. Unstetigkeitsstellen in einem bewegten Kontinuum. Z. Angew. Math. Mech. 21, 152–161 (1941).

The author in large part formally redevelops the method of discontinuities originally due to Hugoniot for certain

one dimensional problems in the motion of elastic media and deals directly with the defining dynamical, kinematical and elastic equations of the problem [cf. J. Hadamard, Leçons sur la propagation des ondes et les équations de l'hydrodynamique, Hermann et Cie, Paris, 1903]. The contribution would seem to be in the exposition of the physical situation and perhaps in a few formal invariants of passage through a discontinuation surface. Application is made to the propagation of the stress wave when one end of a uniform rod moving axially is suddenly brought to rest. The mathematical remarks linking this method with that of the method of characteristics of the governing hyperbolic differential equation are known as well as the conclusion that when Hooke's law is valid discontinuities are propagated with constant velocity (namely, that of sound).

D. G. Bourgin (Urbana, Ill.).

Bechmann, R. Elastische Schwingungen eines anisotropen Körpers von der Form eines rechtwinkligen Parallelepipeds. Z. Phys. 117, 180–197 (1941). [MF 6844]

By differentiating and adding the usual equations of motion

$$\rho\frac{\partial^2 u}{\partial t^2}=-\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right),\quad \text{etc.},$$

we may obtain six equations with the second-order time-derivatives of the strain components on the left. This is essentially what the author does, but certain terms are dropped from the final equations without adequate reasons being given. The following are typical of the six equations, thus simplified:

$$(*)\qquad \rho\frac{\partial^2 x_x}{\partial t^2}=-\frac{\partial^2 X_x}{\partial x^2},\quad \rho\frac{\partial^2 y_z}{\partial t^2}=-\left(\frac{\partial^2 Y_z}{\partial y^2}+\frac{\partial^2 Y_z}{\partial z^2}\right).$$

($x_x, y_y, z_z, y_z, z_x, x_y$ are components of strain.) It is essential for the success of the method that mixed derivatives of the type $\partial^2/\partial y\partial z$ should not occur. The equations of compatibility are mentioned, but play no part in the argument.

By the usual linear stress-strain relations for an anisotropic body, the equations (*) may be expressed either wholly in terms of stress or wholly in terms of strain. The former is more suitable when the boundary conditions are in terms of stress. The solutions for a free rectangular parallelepiped are of the form

$$X_x=E\sin k_1(x-\tfrac{1}{2}l_x)\sin k_2(y-\tfrac{1}{2}l_y)\sin k_3(z-\tfrac{1}{2}l_z)e^{i\omega t},$$

where $k_1=n_1\pi/l_x$, $k_2=n_2\pi/l_y$, $k_3=n_3\pi/l_z$, the faces of the parallelepiped being $x=\pm l_x$, $y=\pm l_y$, $z=\pm l_z$. The other stress components are of the same form, with other constants instead of E. When these expressions are substituted in (*), we get a determinantal equation of the sixth degree (involving the k's) for the characteristic frequencies (ω). The corresponding displacements are easily found. When some of the edges are short, or when there is special symmetry, the determinantal equation simplifies. The method is applied to a bar, a plate and an isotropic body.

J. L. Synge (Toronto, Ont.).

Southwell, R. V. Some practically important stress-systems in solids of revolution. Proc. Roy. Soc. London. Ser. A. 180, 367–396 (1942). [MF 7039]

Saint-Venant's semi-inverse method is used in order to obtain stress distributions in complete solids of revolution and in incomplete tores. Using cylindrical coordinates r, θ, Z, the author studies stress systems in which the stress components either are independent of θ or have $\sin\theta$ or $\cos\theta$ as a factor. The cases treated are: (1) torsional stresses in an incomplete tore, (2) torsion of a shaft of varying circular cross-section, (3) symmetrical strain in a solid of revolution, (4) flexural stresses in an incomplete tore, (5) shearing stresses in a toroidal hook. With the exception of case (1), the solutions obtained are illustrated by simple examples. A comparison of the results with earlier solutions concludes the paper.

W. Prager (Providence, R. I.).

Pflanz, Erwin. Untersuchungen über die Druckverteilung unter belasteten Balken auf nachgiebiger Unterlage. Ing.-Arch. 12, 201–221 (1941). [MF 6423]

In most technical treatments it has been the custom to assume the relation

$$(1)\qquad w(x)=q(x)/k$$

between the deflection w of a beam and the reaction q of the supporting medium. When this relation is combined with the differential equation of the theory of beams as used by engineers no particular mathematical difficulties are encountered. The relation (1) however is not entirely satisfactory as it expresses a purely localized relation between q and w and leads to a discontinuity in the deflection of the supporting medium at the ends of the beam. To take into account the fact that q acting at a point ξ influences the value of w at a point x different from ξ, K. Weighardt [Z. Angew. Math. Mech. 2, 165–184 (1922)] proposed to replace (1) by the relation

$$(2)\qquad w(x)=\int_{-L}^{L}K(|x-\xi|)q(\xi)d\xi.$$

He showed that, if $K(r)$ is taken to be of the form $Ce^{-\lambda r}$, the problem of determining $w(x)$ and $q(x)$ cannot be solved, as the number of boundary conditions to be satisfied exceeds the number of constants of integration to be determined. It is shown in the present paper that the problem can be solved if the loading of the beam satisfies a certain condition and the function K has the exponential form given above. It is also shown that the problem cannot in general be solved when $K(r)$ is a polynomial in r but that under a certain condition of loading it can be solved. It is pointed out that Weighardt, in the paper previously referred to, obtained a solution for the case where $K(r)$ is of the exponential form and there is a rectangular distribution of load by using a more exact theory for the beam than that expressed by the differential equation ordinarily used in engineering practice, thus indicating that the source of the difficulties encountered is to be found in the approximations involved in this differential equation and not in the choice of the function $K(r)$.

H. W. March (Madison, Wis.).

Higuchi, Seiichi. Stresses in a semi-infinite strip under the forces applied at its end. Tech. Reports Tôhoku Imp. Univ. 13, 289–301 (1941). [MF 6983]

Expressions purporting to specify the exact stress distribution in a loaded semi-infinite elastic strip are presented and evaluated. The solution given is defective in that the stresses do not satisfy the equation of compatibility.

F. B. Hildebrand (Cambridge, Mass.).

Higuchi, Seiichi and Iinuma, Kazukiyo. Figures and tables for displacement, bending moment and shearing force of a cantilever at the time of stationary oscillation. Tech. Reports Tôhoku Imp. Univ. 13, 302–312 (1941). [MF 6984]

Higuchi, Seiichi. A method of solving two dimensional elastic problems and its applications. Tech. Reports Tôhoku Imp. Univ. 13, 313–333 (1941). [MF 6985]

Procedures given by Kolossoff [Z. Math. Phys. 62, 384–409 (1914)], and credited without explicit reference in this paper to Yokota, are applied to the treatment of certain previously solved problems in the theory of plane stress. *F. B. Hildebrand* (Cambridge, Mass.).

Stevenson, A. C. On the equilibrium of plates. Philos. Mag. (7) 33, 639–661 (1942). [MF 7227]

The author utilizes certain complex combinations of stress resultants and stress couples for the purpose of simplifying the usual treatment of the equilibrium equations of thick and thin plates. The stress and couple resultants are expressed in terms of complex potentials. The use of complex potentials is illustrated by a consideration of an elliptical and of one type of triangular plates. *I. S. Sokolnikoff.*

Ratzersdorfer, J. Rectangular plates with stiffeners. The buckling of simply supported plates under compressive stress. Aircraft Engrg. 14, 260–263 (1942). [MF 7293]

The buckling of thin stiffened rectangular plates with simply supported edges, under the action of uniformly distributed compressive forces along two parallel edges, is investigated by an exact analysis. General formulas for the determination of the critical buckling stress, for any number of equally spaced equal stiffeners in the longitudinal or transverse direction, are given in convenient form, and are evaluated in several special cases. *F. B. Hildebrand.*

Hencky, H. Determining critical states of equilibrium of plates and shells under initial stress. J. Appl. Mech. 9, A-27–A-30 (1942). [MF 6266]

An elastic body is in equilibrium in state 1 under boundary stresses and body forces. It is brought to a second state 2, not necessarily one of equilibrium. The principle of energy (adiabatic) is applied to compare states 1 and 2 by integration through the transition from one state to the other. Certain approximations are made, based on the assumption that the rotations are much greater than the strains. The principal equation obtained in this way may be written

$$I_1(P, \Delta\omega, \Delta u)+\Delta T+I_2(\sigma, \Delta\omega)=I_3(\Delta\epsilon),$$

where ΔT is the kinetic energy in state 2, I_1 is a surface integral and I_2, I_3 volume integrals involving the variables indicated; P stands for the surface stresses and σ for the internal stresses in state 1; Δu, $\Delta\omega$, $\Delta\epsilon$ stand for the increments in displacement, rotation and strain in passing from state 1 to state 2. The principal equation is applied to the problem of the stability of a cylindrical shell. A displacement of the middle surface is assumed, and the corresponding values of Δu, $\Delta\omega$, $\Delta\epsilon$ are calculated. Then ΔT is put equal to zero in the principal equation, and this equation gives the critical pressure on the shell. [There are several points in the deduction of the principal equation that are by no means clear to the reviewer.] *J. L. Synge.*

von Kármán, Theodore and Tsien, Hsue-Shen. The buckling of thin cylindrical shells under axial compression. J. Aeronaut. Sci. 8, 303–312 (1941). [MF 5329]

This paper is devoted to the solution of the problem stated in the title under the same general assumptions as were made in an earlier paper by the same authors on buckling of the spherical shell [J. Aeronaut. Sci. 7, 43–50 (1939); these Rev. 2, 175]. [See also a paper by v. Kármán, Dunn and Tsien [J. Aeronaut. Sci. 7, 276–289 (1940); these Rev. 2, 176] and a paper by K. Friedrichs [Theodore von Kármán Anniversary Volume, California Institute of Technology, Pasadena, 1941, pp. 258–272; these Rev. 3, 31].] The essentially new idea in this as in the earlier papers is that it is possible to explain why thin shells buckle at a much lower pressure than that predicted by the linear theory of buckling by considering the effect of certain nonlinear terms; even the quantitative results of the nonlinear theory for the spherical shell were found to be in quite good accord with experiment. The cylindrical shell is a much more difficult case than that of the spherical shell (because of a lack of symmetry in the buckled state) so that the authors restrict themselves in the main to qualitative rather than quantitative comparison with experiment. Solutions are obtained by an energy method. *J. J. Stoker.*

Tsien, H. S. Buckling of a column with non-linear lateral supports. J. Aeronaut. Sci. 9, 119–132 (1942). [MF 6246]

The author makes a detailed study of a special nonlinear buckling problem. The results obtained are significant because they help to explain and to promote understanding of the much more difficult problems concerning buckling of thin shells in which a large discrepancy between the linear theory and the experiments has been found [J. Aeronaut. Sci. 7, 43–50 (1939); 7, 276–289 (1940); these Rev. 2, 175, 176]. As indicated in the title, the elastic system considered is that of a column subjected to compressive forces P at the ends and supported laterally by a number of springs with nonlinear characteristics. In the numerical examples the springs are assumed to be such that eventually the spring force decreases and even becomes a tension for large enough compressive displacements of the spring, a rather artificial sort of spring. For the column itself the linear theory is retained. The results are quite novel. To begin with, it is important to distinguish between two different buckling problems: (1) that in which the load P is prescribed; (2) that in which the end shortening ϵ of the column is prescribed. In the first case the most striking result is that P decreases with increase of the lateral deflection δ once buckling has begun (that is, once the straight state becomes instable). If then P is plotted as a function of the end shortening ϵ it is found that P is not a single-valued function of ϵ and that there are values of ϵ for which P is smaller in the deflected than in the undeflected state. Since the conditions under which tests are made in the laboratory correspond more to those of problem (2) than to those of (1), this would account for the observed jump from the unbuckled state to a buckled state at a lower pressure P. The criterion for the most probable state would be, one might expect, that for which the strain energy is a minimum. There is, however, a difficulty due to the fact that both the unbuckled straight state and the buckled state (for the same ϵ) are states of stable equilibrium, so that it is not clear why there should be a jump from one to the other. The author is able to show in his numerical case that the energy at the "saddle point" between the minima is only slightly higher than the one minimum, so that a small disturbance would be sufficient to carry the column from one stable state to another at a lower energy level. The author also investigates the effect of applying the load P to the column through springs (thus simulating the elasticity of a testing machine) and the effect of slight initial curvature of the column. *J. J. Stoker* (New York, N. Y.).

*Goodier, J. N. **The buckling of compressed bars by torsion and flexure.** Cornell University Engineering Experiment Station, Bulletin no. 27. Ithaca, N. Y., 1941. 32 pp. $.50.

*Goodier, J. N. **Flexural-torsional buckling of bars of open section, under bending, eccentric thrust of torsional loads.** Cornell University Engineering Experiment Station, Bulletin no. 28. Ithaca, N. Y., 1942. 16 pp. $.30.

These papers are contributions to the elastic stability theory of straight bars. The differential equations of the theory are constructed so as to include (1) the influence of nonuniform torsion on the stress resultant strain relations, and (2) the effect of the torsional couple of the axial initial stresses which arises during a torsional deformation of the bar. Inclusion of these effects makes possible, for instance, the prediction of the critical axial loads of beams with thin-walled open cross section which may fail either by bending or by twisting, the type of failure depending on the characteristics of the cross section.

In the first paper the author re-examines the theory for buckling due only to axial compression which had previously been treated by H. Wagner [Technische Hochschule, Danzig, 25th Anniversary Number, 1929, pp. 329–343; translated as Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 807], by R. Kappus [Luftfahrtforschung **14**, 444–457 (1937)] and by E. E. Lundquist and C. M. Fligg [Tech. Reports. Nat. Adv. Comm. Aeronaut., no. **582**, Government Printing Office, Washington, D. C., 1937]. It is indicated that a simplification of the theory is accomplished by specifying the movement of a cross section by the displacement of the flexural center and a rotation instead of by the displacement of the centroid and a rotation. Not previously given results which are included in the paper concern the problem of a stiffener attached to a flexible sheet, buckling with an enforced axis of rotation and a discussion of the effects of initial imperfections of the bar.

In the second paper the theory is extended to bars with an initial state of stress due to axial end loads and end bending and twisting moments. There is then obtained a generalization of the theory of the lateral instability of deep beams which had been developed mainly by Michell, Prandtl, H. Reissner and Timoshenko [see S. Timoshenko, Theory of Elastic Stability, McGraw-Hill, New York, 1936]. *E. Reissner* (Cambridge, Mass.).

Goodier, J. N. Torsional and flexural buckling of bars of thin-walled open section under compressive and bending loads. J. Appl. Mech. **9**, A-103–A-107 (1942). [MF 7124]

This is a condensed version of the two papers reviewed above. *E. Reissner* (Cambridge, Mass.).

*Sturm, Rolland G. **A Study of the Collapsing Pressure of Thin-walled Cylinders.** University of Illinois Bulletin, v. 39, no. 12=Engineering Experiment Station Bulletin Series no. 329. University of Illinois, Urbana, Ill., 1941. 77 pp. $.80.

The purposes of this study are (a) to analyze the elastic behavior of their circular cylindrical shells subjected to uniform external pressure, and to determine the pressure at which such shells collapse for simply supported and for fixed edges; extensions in the analysis are made for plastic behavior of the material, for "out-of-roundness" of the cylinder, and for stiffening effects of ring stiffeners; (b) to study experimentally the behavior of thin-walled tubes under uniform external pressures, for comparison with the results of the theoretical analysis. *E. Reissner.*

Schubert, G. Über Effekte zweiter Ordnung bei Biegung und Torsion dünnwandiger Rohre elliptischen Querschnitts. Ing.-Arch. **12**, 53–63 (1941). [MF 6862]

The author studies the secondary stresses occurring in the bending of tubes of elliptical cross section. For thin-walled tubes those stresses producing a flattening of the cross section must be taken into account. The degree of flattening is computed by means of Castigliano's principle. The case of torsion is treated in a similar way. The results are applied to the theory of the Bourdon manometer. *W. Prager* (Providence, R. I.).

Higgins, Thomas James. Remarks on a method for solving the torsion problem. Nat. Math. Mag. **17**, 3–7 (1942). [MF 7234]

The remarks deal with a discussion of the derivation of the complex torsion function $\Omega=\phi+i\psi$ for right prisms whose cross sections have boundaries which can be expressed as finite sums of the form

$$z\bar{z}=A_0+\tfrac{1}{2}\sum(A_n z^n+\bar{A}_n\bar{z}^n).$$

These include the equilateral triangle, the circle, the circular annulus, the circle notched by a second circle with its center on the perimeter of the given prism and a section not hitherto discussed. The latter is a prism bounded by the hyperbolas

$$[x^2(\sqrt{2}+1)-y^2(\sqrt{2}-1)-a^2]\times[y^2(\sqrt{2}+1)-x^2(\sqrt{2}-1)-b^2]=0.$$

D. L. Holl (Ames, Iowa).

Dick, J. The transverse vibrations of a helical spring with pinned ends and no axial load. Philos. Mag. (7) **33**, 513–519 (1942). [MF 7088]

The paper contains calculations leading to formulas for the determination of frequencies of possible modes of transverse vibration of a helical spring with pinned ends and free of axial loads. *I. S. Sokolnikoff* (Madison, Wis.).

Opatowski, I. A theory of brakes, an example of a theoretical study of wear. J. Franklin Inst. **234**, 239–249 (1942). [MF 7105]

The paper contains an essentially kinematic analysis of the action of brakes, account being taken of the elastic deformation as well as of the wear of the brake shoe lining. The friction developed at a point of contact of brake shoe and drum is supposed to be proportional to the elastic compression of the brake shoe lining. Th. Reye's law that the volume of brake shoe lining worn by friction is proportional to the work done by the friction forces then permits the determination of the wear of the brake shoe lining. The influence of the wear on the brake characteristic can thus be evaluated. *W. Prager* (Providence, R. I.).

Pippard, A. J. S. The geometrical derivation of formulas for laterally loaded struts. Math. Gaz. **26**, 119–129 (1942). [MF 7104]

J. Ratzersdorfer and H. B. Howard have used polar bending moment diagrams in the graphical analysis of structural members under combined axial and transverse loads. The author points out that general formulas concerning the distribution of bending moments can often be deduced from a freehand sketch of the polar diagram. *W. Prager* (Providence, R. I.).